

This paper works from the Results of the IAP lecture “Financial Derivatives,” and the associated appendix “Pricing Risky Options with Nearly Continuous Hedging,” by Martin Bazant. The Lecture and Appendix may be found online at <http://www-math.mit.edu/bazant/teach/IAP00-finance>.

The paper is divided into 3 sections. The first section checks the calculations of the Appendix, the second makes an extension to the case of a unique risk-free interest rate, and the third considers the case of lognormal price dynamics.

## 1 Check of the Results of the Appendix

This section carries out all the steps of the calculations for the results of the Appendix. We start by re-stating Eq. (7),

$$(1) \quad w(x, t) = \langle w \rangle + \phi^*(x - \langle x \rangle)$$

This equation is a statement of the dynamic hedging strategy used to minimize the quadratic risk of the hedged portfolio. Written out with full notation, we have

$$(2) \quad w(x, t) = \int w(x', t + \delta t) p(x', t + \delta t | x, t) dx' \\ + \phi^*(x, t) \left( x - \int x' p(x', t + \delta t | x, t) dx' \right)$$

Note that the arguments  $(x, t)$  of  $\phi^*$  must match the arguments of  $w$  on the left-hand side, since the amount  $\phi^*$  is chosen at time  $t$ , to hedge price movements at the later time  $t + \delta t$ .

Next using the “Markov” relation for the transition probability  $p$ , stated as Eq. (1) of the Appendix, we have

$$(3) \quad w(x, t) = \int w(x + \delta x, t + \delta t) p(\delta x, \delta t) d(\delta x) \\ + \phi^*(x, t) \left( x - \int (x + \delta x) p(\delta x, \delta t) d(\delta x) \right)$$

where  $\delta x = x' - x$ ,  $dx' = d(\delta x)$ , and the (so far unstated) limits of integration remain unchanged at  $(-\infty, \infty)$ .

We also note that  $x$  is a constant in the integrals of (3), since it represents the price at time  $t$ . We use this to write

$$(4) \quad w(x, t) = \int w(x + \delta x, t + \delta t) p(\delta x, \delta t) d(\delta x) - \phi^*(x, t) (\mu \delta t)$$

This equation is the full statement of the final step of Eq. (7) from the Appendix.

In considering Eq. (4), we first write out  $\phi^*$  in a different form from the Appendix. The formula there is stated as

$$(5) \quad \phi^* = \frac{\langle xw \rangle - \langle x \rangle \langle w \rangle}{\langle x^2 \rangle - \langle x \rangle^2}$$

where again we note that all expectation are taken over the price at the future time  $t + \delta t$ , given that the price is  $x$  at time  $t$ . We use manipulations similar to those used to obtain Eq. (4) to write

$$(6) \quad \phi^* = \frac{\langle \delta x w \rangle - \langle \delta x \rangle \langle w \rangle}{\text{Var}(\delta x)}$$

To simplify (6), we first change the statement of the moments from the Appendix. Eqs. (4)-(6) there state the form of the de-meaned moments,

$$\begin{aligned}
(7) \quad & \langle \delta x \rangle = \mu \delta t \\
& \langle (\delta x - \mu \delta t)^2 \rangle = \sigma^2 \delta t \\
& \langle (\delta x - \mu \delta t)^3 \rangle = \sigma^3 \lambda_3 \delta t^{3/2} \\
& \langle (\delta x - \mu \delta t)^4 \rangle = \sigma^4 (\lambda_4 + 3) \delta t^2
\end{aligned}$$

The moments without de-meaning are given by

$$\begin{aligned}
(8) \quad & \langle \delta x \rangle = \mu \delta t \\
& \langle \delta x^2 \rangle = \sigma^2 \delta t + \mu^2 \delta t^2 \\
& \langle \delta x^3 \rangle = \sigma^3 \lambda_3 \delta t^{3/2} + 3\mu\sigma^2 \delta t^2 + \mu^3 \delta t^3 \\
& \langle \delta x^4 \rangle = \sigma^4 (\lambda_4 + 3) \delta t^2 + 4\mu\sigma^3 \lambda_3 \delta t^{5/2} + 6\mu^2 \sigma^2 \delta t^3 + \mu^4 \delta t^4
\end{aligned}$$

We combine terms in the numerator, and use these refined moments, to write out  $\phi^*$  in its full form as

$$(9) \quad \phi^*(x, t) = \frac{1}{\sigma^2 \delta t} \left[ \int (\delta x - \mu \delta t) w(x + \delta x, t + \delta t) p(\delta x, \delta t) d(\delta x) \right]$$

We next approximate  $\phi^*$  to  $O(\delta t)$  as follows. We write out the Taylor expansion for  $w$ ,

$$\begin{aligned}
(10) \quad & w(x + \delta x, t + \delta t) \approx w \\
& + \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial t} \delta t \\
& + \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \delta x^2 + \frac{1}{2} \frac{\partial^2 w}{\partial t^2} \delta t^2 + \frac{\partial^2 w}{\partial x \partial t} \delta x \delta t \\
& + \frac{1}{6} \frac{\partial^3 w}{\partial x^3} \delta x^3 + \frac{1}{6} \frac{\partial^3 w}{\partial t^3} \delta t^3 + \frac{1}{2} \frac{\partial^3 w}{\partial x \partial t^2} \delta x \delta t^2 + \frac{1}{2} \frac{\partial^3 w}{\partial x^2 \partial t} \delta x^2 \delta t
\end{aligned}$$

where all terms on the left-hand side are evaluated at  $(x, t)$ . We replace  $w$  by this expansion in (9), and integrate term by term, to obtain

$$\begin{aligned}
(11) \quad & \phi^*(x, t) = \frac{1}{\sigma^2 \delta t} [ \\
& 0 \\
& + \frac{\partial w}{\partial x} (\sigma^2 \delta t) \\
& + 0 \\
& + \frac{\partial^2 w}{\partial x^2} \left( \frac{\sigma^3 \lambda_3}{2} \delta t^{3/2} + \mu \sigma^2 \delta t^2 \right) \\
& + 0 \\
& + \frac{\partial^2 w}{\partial x \partial t} (\sigma^2 \delta t^2) \\
& + \frac{\partial^3 w}{\partial x^3} \left( \frac{\sigma^4 (\lambda_4 + 3)}{6} \delta t^2 \right) \\
& + O(\delta t^{5/2}) ]
\end{aligned}$$

We group by order- $\delta t$ , which gives the expression

$$\begin{aligned}
(12) \quad \phi^*(x, t) &= \frac{\partial w}{\partial x} \\
&+ (\delta t^{1/2}) \left[ \frac{\sigma \lambda_3}{2} \left( \frac{\partial^2 w}{\partial x^2} \right) \right] \\
&+ (\delta t) \left[ \mu \left( \frac{\partial^2 w}{\partial x^2} \right) + \frac{\partial^2 w}{\partial x \partial t} + \frac{\sigma^2 (\lambda_4 + 3)}{6} \left( \frac{\partial^3 w}{\partial x^3} \right) \right] \\
&+ O(\delta t^{3/2})
\end{aligned}$$

This equation corresponds to Eq. (9) from the Appendix. The expressions differ in the  $O(\delta t)$  term.

Eq. (12) gives an expression which may be substituted into the dynamic hedging equation (4) to get an  $O(\delta t^2)$  approximation for the second term there. We next carry out a similar procedure for the first term in (4), that is, the expectation of  $w$ . This time we keep terms to order  $\delta t^2$ . The result is

$$\begin{aligned}
(13) \quad &\int w(x + \delta x, t + \delta t) p(\delta x, \delta t) d(\delta x) \\
&= w \\
&+ (\delta t) \left[ \mu \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} \right] \\
&+ (\delta t)^{3/2} \left[ \frac{\sigma^3 \lambda_3}{6} \frac{\partial^3 w}{\partial x^3} \right] \\
&+ (\delta t)^2 \left[ \frac{\mu^2}{2} \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w}{\partial t^2} + \mu \frac{\partial^2 w}{\partial x \partial t} + \frac{\mu \sigma^2}{2} \frac{\partial^3 w}{\partial x^3} + \frac{\sigma^2}{2} \frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\sigma^4 (\lambda_4 + 3)}{24} \frac{\partial^4 w}{\partial x^4} \right] \\
&+ O(\delta t^{5/2})
\end{aligned}$$

We combine the approximations (12) and (13) to re-write the dynamic hedging equation (4). The result is

$$\begin{aligned}
(14) \quad 0 &= \left[ \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} \right] \\
&+ (\delta t^{1/2}) \left[ \frac{\sigma^3 \lambda_3}{6} \frac{\partial^3 w}{\partial x^3} - \frac{\mu \sigma \lambda_3}{2} \frac{\partial^2 w}{\partial x^2} \right] \\
&+ (\delta t) \left[ -\frac{\mu^2}{2} \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w}{\partial t^2} + \frac{\sigma^2}{2} \frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\sigma^4 (\lambda_4 + 3)}{24} \frac{\partial^4 w}{\partial x^4} - \frac{\mu \sigma^2 \lambda_4}{6} \frac{\partial^3 w}{\partial x^3} \right] \\
&+ O(\delta t^{3/2})
\end{aligned}$$

Eq. (14) expresses the Black-Scholes equation plus corrections to  $O(\delta t)$ , for models whose dynamics may be described by the moments in (7) and (8), and a zero interest rate. We next replace all time derivatives at  $O(\delta t)$  by spatial derivatives, by the following two-step procedure. We first combine two of the terms:

$$\begin{aligned}
(15) \quad \frac{1}{2} \frac{\partial^2 w}{\partial t^2} + \frac{\sigma^2}{2} \frac{\partial^3 w}{\partial x^2 \partial t} &= -\frac{1}{2} \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial t} \left[ \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} + O(\delta t^{1/2}) \right] \\
&= -\frac{1}{2} \frac{\partial^2 w}{\partial t^2} + O(\delta t^{1/2})
\end{aligned}$$

We next approximate the right-hand side of (15) similarly,

$$\begin{aligned}
(16) \quad \frac{\partial w}{\partial t} &= -\frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} + O(\delta t^{1/2}) \\
\frac{1}{2} \frac{\partial^2 w}{\partial t^2} &= -\frac{\sigma^2}{4} \frac{\partial^2}{\partial x^2} \frac{\partial w}{\partial t} + O(\delta t^{1/2}) \\
&= \frac{\sigma^4}{8} \frac{\partial^4 w}{\partial x^4} + O(\delta t^{1/2})
\end{aligned}$$

The terms at  $O(\delta t^{1/2})$  in (15) and (16) enter only at  $O(\delta t^{3/2})$  in the hedging equation (14), thus allowing for the simplified form

$$\begin{aligned}
(17) \quad 0 = & \left[ \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} \right] \\
& + (\lambda_3 \delta t^{1/2}) \left[ \frac{\sigma^3}{6} \frac{\partial^3 w}{\partial x^3} - \frac{\mu \sigma}{2} \frac{\partial^2 w}{\partial x^2} \right] \\
& + (\delta t) \left[ \lambda_4 \left( \frac{\sigma^4}{24} \frac{\partial^4 w}{\partial x^4} - \frac{\mu \sigma^2}{6} \frac{\partial^3 w}{\partial x^3} \right) - \frac{\mu^2}{2} \frac{\partial^2 w}{\partial x^2} \right] \\
& + O(\delta t^{3/2})
\end{aligned}$$

Eq. (17) nearly presents a form where the size of each new order- $\delta t$  correction term depends on the next highest cumulant of  $\delta x$ . An alternate form is to group by  $w$ -derivative,

$$\begin{aligned}
(18) \quad 0 = & \frac{\partial w}{\partial t} \\
& + \frac{\partial^2 w}{\partial x^2} \left[ \frac{\sigma^2}{2} - \frac{\mu \sigma \lambda_3}{2} \delta t^{1/2} - \frac{\mu^2}{2} \right] \\
& + \frac{\partial^3 w}{\partial x^3} \left[ \frac{\sigma^3 \lambda_3}{6} \delta t^{1/2} - \frac{\mu \sigma^2 \lambda_4}{6} \delta t \right] \\
& + \frac{\partial^4 w}{\partial x^4} \left[ \frac{\sigma^4 \lambda_4}{24} \delta t \right] \\
& + O(\delta t^{3/2})
\end{aligned}$$

## 2 Nonzero, Unique Risk-Free Interest Rate

The Appendix and Section (1) were worked out in the absence of interest-rate effects. This section extends the previous formulations to include a nonzero, unique risk-free interest rate.

It is an oversimplification, but a useful first approximation, to assume a unique risk-free rate  $r$ . Such a rate was derived by Black and Scholes, since by their model it was possible to completely hedge away all risk, and thereby create a riskless investment. If it is in general impossible to eliminate all risk, as assumed here, then the rate for borrowers and lenders should differ. The resulting consequences remain for future work.

Re-stating the dynamic hedging equation (2), we have

$$\begin{aligned}
(19) \quad w(x, t) - \phi^*(x, t)x \\
= \int w(x', t + \delta t) p(x', t + \delta t | x, t) dx' - \phi^*(x, t) \int x' p(x', t + \delta t | x, t) dx'
\end{aligned}$$

This equation states that the current value of the hedge portfolio should be equal to its expected value one time step in the future. In the presence of an interest rate  $r$ , the net value of the hedge portfolio must be borrowed, and the resulting liability must earn interest. With this modification, we have

$$\begin{aligned}
(20) \quad e^{r\delta t} (w(x, t) - \phi^*(x, t)x) \\
= \int w(x', t + \delta t) p(x', t + \delta t | x, t) dx' - \phi^*(x, t) \int x' p(x', t + \delta t | x, t) dx'
\end{aligned}$$

Re-arranging terms, and again using the Markov relation for the transition probability  $p$ , leads to

$$\begin{aligned}
(21) \quad w(x, t) = e^{-r\delta t} \int w(x + \delta x, t + \delta t) p(\delta x, \delta t) d(\delta x) \\
+ \phi^*(x, t) [x(1 - e^{-r\delta t}) - e^{-r\delta t} \mu \delta t]
\end{aligned}$$

Note that this expression reduces to Eq. (4) from Section (1) in the case  $r = 0$ . Also note that  $\phi^*$  remains unchanged from Section (1).  $\phi^*$  is still chosen to minimize the variance of the hedged portfolio, as detailed in Section 4 of the Lecture. Hence we still use Eq. (12) to approximate  $\phi^*$  to  $O(\delta t)$ .

We also retain Eq. (13) to approximate the expectation for  $w$ , but now we multiply in the exponential for the risk-free growth rate. Since we wish to approximate this term to  $O(\delta t^2)$ , we use the relation

$$(22) \quad e^{-r\delta t} \approx 1 - r \delta t + \frac{r^2}{2} \delta t^2$$

We multiply (22) into the relation (13) for  $w$ , and group terms by order- $\delta t$ , to obtain

$$(23) \quad \begin{aligned} & e^{-r\delta t} \int w(x + \delta x, t + \delta t) p(\delta x, \delta t) d(\delta x) \\ &= w \\ &+ (\delta t) \left[ \mu \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} - r w \right] \\ &+ (\delta t^{3/2}) \left[ \frac{\sigma^3 \lambda_3}{6} \frac{\partial^3 w}{\partial x^3} \right] \\ &+ (\delta t^2) \left[ \frac{\mu^2}{2} \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w}{\partial t^2} + \mu \frac{\partial^2 w}{\partial x \partial t} + \frac{\mu \sigma^2}{2} \frac{\partial^3 w}{\partial x^3} + \frac{\sigma^2}{2} \frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\sigma^4 (\lambda_4 + 3)}{24} \frac{\partial^4 w}{\partial x^4} \right] \\ &- (r \delta t^2) \left[ \mu \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} - \frac{r}{2} w \right] \\ &+ O(\delta t^{5/2}) \end{aligned}$$

We leave the  $O(\delta t^2)$  terms split out, because of a simplification that will be made later.

At the second step, we again use the approximate relation (22) for the exponential, along with (12) for  $\phi^*$ , to approximate the second term of the hedging equation (21) to  $O(\delta t^2)$ .

$$(24) \quad \begin{aligned} & \phi^*(x, t) [x(1 - e^{-r\delta t}) - e^{-r\delta t}(\mu\delta t)] \\ &= (\delta t) \left[ (rx - \mu) \frac{\partial w}{\partial x} \right] \\ &+ (\delta t^{3/2}) \left[ \frac{(rx - \mu)\sigma\lambda_3}{2} \frac{\partial^2 w}{\partial x^2} \right] \\ &+ (\delta t^2) \left[ (rx - \mu)\mu \frac{\partial^2 w}{\partial x^2} + (rx - \mu) \frac{\partial^2 w}{\partial x \partial t} + \frac{(rx - \mu)\sigma^2(\lambda_4 + 3)}{6} \frac{\partial^3 w}{\partial x^3} + \left( \mu r - \frac{r^2 x}{2} \right) \frac{\partial w}{\partial x} \right] \\ &+ O(\delta t^{5/2}) \end{aligned}$$

We next substitute (23) and (24) into the hedging equation (21). After some manipulations, we have

$$(25) \quad \begin{aligned} 0 &= \left[ rx \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} - r w \right] \\ &+ (\delta t^{1/2}) \left[ \frac{\sigma^3 \lambda_3}{6} \frac{\partial^3 w}{\partial x^3} + \frac{(rx - \mu)\sigma\lambda_3}{2} \frac{\partial^2 w}{\partial x^2} \right] \\ &+ (\delta t) \left[ \left( rx - \frac{\mu}{2} \right) \mu \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w}{\partial t^2} + rx \frac{\partial^2 w}{\partial x \partial t} + \frac{(rx - \mu)\sigma^2 \lambda_4}{6} \frac{\partial^3 w}{\partial x^3} + \frac{rx\sigma^2}{2} \frac{\partial^3 w}{\partial x^3} + \frac{\sigma^2}{2} \frac{\partial^3 w}{\partial x^2 \partial t} \right. \\ &\quad \left. + \frac{\sigma^4 (\lambda_4 + 3)}{24} \frac{\partial^4 w}{\partial x^4} \right] \\ &- (r\delta t) \left[ \frac{rx}{2} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} - \frac{r}{2} w \right] \\ &+ O(\delta t^{3/2}) \end{aligned}$$

We simplify the  $O(\delta t)$  terms, using a procedure similar to that used at the end of Section (1). We write out the relation

$$(26) \quad \frac{\partial w}{\partial t} = rw - rx \frac{\partial w}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} + O(\delta t^{1/2})$$

We use this to make the substitutions

$$(27) \quad (-r\delta t) \left[ \frac{rx}{2} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} - \frac{r}{2} w \right] = (\delta t) \left[ -\frac{r^2}{2} w + \frac{r^2 x}{2} \frac{\partial w}{\partial x} \right] + O(\delta t^{3/2})$$

$$(28) \quad \begin{aligned} \frac{1}{2} \frac{\partial^2 w}{\partial t^2} + rx \frac{\partial^2 w}{\partial x \partial t} + \frac{\sigma^2}{2} \frac{\partial^3 w}{\partial x^2 \partial t} &= w \left( \frac{r^2}{2} \right) \\ &+ \frac{\partial w}{\partial x} \left( -\frac{r^2 x}{2} \right) \\ &+ \frac{\partial^2 w}{\partial x^2} \left( -\frac{r\sigma^2}{2} - \frac{r^2 x^2}{2} \right) \\ &+ \frac{\partial^3 w}{\partial x^3} \left( -\frac{r\sigma^2 x}{2} \right) \\ &+ \frac{\partial^4 w}{\partial x^4} \left( -\frac{\sigma^4}{8} \right) \end{aligned}$$

Using these substitutions, we re-write (25) as

$$(29) \quad \begin{aligned} 0 &= \left[ \frac{\partial w}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} - rw + rx \frac{\partial w}{\partial x} \right] \\ &+ (\lambda_3 \delta t^{1/2}) \left[ \frac{\sigma^3}{6} \frac{\partial^3 w}{\partial x^3} + \frac{(rx - \mu)\sigma}{2} \frac{\partial^2 w}{\partial x^2} \right] \\ &+ (\delta t) \left[ \lambda_4 \left( \frac{\sigma_4}{24} \frac{\partial^4 w}{\partial x^4} + \frac{(rx - \mu)\sigma^2}{6} \frac{\partial^3 w}{\partial x^3} \right) + \left( \left( rx - \frac{\mu}{2} \right) \mu - \frac{r\sigma^2}{2} - \frac{r^2 x^2}{2} \right) \frac{\partial^2 w}{\partial x^2} \right] \\ &+ O(\delta t^{3/2}) \end{aligned}$$

Importantly, we note that (29) reduces to Eq. (17) from Section (1), if  $r = 0$ . Once again, we nearly have an expression where the correction term at each higher order of  $\delta t$  depends on the next highest cumulant of  $\delta x$ . We show (29) with terms grouped by  $w$ -derivative,

$$(30) \quad \begin{aligned} 0 &= \left[ \frac{\partial w}{\partial t} - rw + rx \frac{\partial w}{\partial x} \right] \\ &+ \frac{\partial^2 w}{\partial x^2} \left[ \frac{\sigma^2}{2} + \delta t^{1/2} \left( \frac{(rx - \mu)\sigma\lambda_3}{2} \right) + \delta t \left( \left( rx - \frac{\mu}{2} \right) \mu - \frac{r\sigma^2}{2} - \frac{r^2 x^2}{2} \right) \right] \\ &+ \frac{\partial^3 w}{\partial x^3} \left[ \delta t^{1/2} \left( \frac{\sigma^3 \lambda_3}{6} \right) + \delta t \left( \frac{\sigma^2 \lambda_4 (rx - \mu)}{6} \right) \right] \\ &+ \frac{\partial^4 w}{\partial x^4} \left[ \delta t \left( \frac{\sigma^4 \lambda_4}{24} \right) \right] \\ &+ O(\delta t^{3/2}) \end{aligned}$$