Experiments with Cache-Oblivious Matrix Multiplication for 18.335

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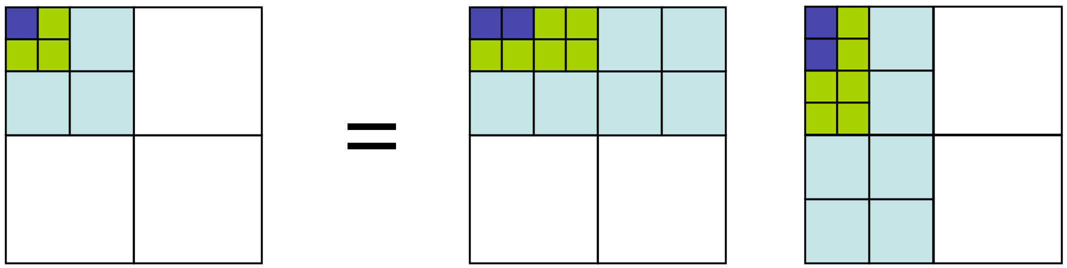
platform: 2.66GHz Intel Core 2 Duo,
GNU/Linux + gcc 4.1.2 (-O3) (64-bit), double precision

(optimal) Cache-Oblivious Matrix Multiply



C $\qquad$ A $\qquad$ B

$m \times p$ $\qquad$ $m \times n$ $\qquad$ $n \times p$

divide and conquer:

divide C into 4 blocks

compute block multiply recursively

achieves optimal $\Theta(n^3/\sqrt{Z})$ cache complexity

A little C implementation (~25 lines)

```
/* C = C + AB, where A is m x n, B is n x p, and C is m x p, in
   row-major order.  Actually, the physical size of A, B, and C
   are m x fdA, n x fdB, and m x fdC, but only the first n/p/p
   columns are used, respectively. */
void add_matmul_rec(const double *A, const double *B, double *C,
                    int m, int n, int p, int fdA, int fdB, int fdC)
{
     if (m+n+p <= 48) { /* <= 16x16 matrices "on average" */
          int i, j, k;
          for (i = 0; i < m; ++i)
               for (k = 0; k < p; ++k) {
                    double sum = 0;
                    for (j = 0; j < n; ++j)
                         sum += A[i*fdA +j] * B[j*fdB + k];
                    C[i*fdC + k] += sum;
               }
     }
     else { /* divide and conquer */
          int m2 = m/2, n2 = n/2, p2 = p/2;

          add_matmul_rec(A, B, C, m2, n2, p2, fdA, fdB, fdC);
          add_matmul_rec(A+n2, B+n2*fdB, C, m2, n-n2, p2, fdA, fdB, fdC);

          add_matmul_rec(A, B+p2, C+p2, m2, n2, p-p2, fdA, fdB, fdC);
          add_matmul_rec(A+n2, B+p2+n2*fdB, C+p2, m2, n-n2, p-p2, fdA, fdB, fdC);

          add_matmul_rec(A+m2*fdA, B, C+m2*fdC, m-m2, n2, p2, fdA, fdB, fdC);
          add_matmul_rec(A+m2*fdA+n2, B+n2*fdB, C+m2*fdC, m-m2, n-n2, p2, fdA, fdB, fdC);

          add_matmul_rec(A+m2*fdA, B+p2, C+m2*fdC+p2, m-m2, n2, p-p2, fdA, fdB, fdC);
          add_matmul_rec(A+m2*fdA+n2, B+p2+n2*fdB, C+m2*fdC+p2, m-m2, n-n2, p-p2, fdA, fdB, fdC);
     }
}

void matmul_rec(const double *A, const double *B, double *C,
                int m, int n, int p)
{
     memset(C, 0, sizeof(double) * m*p);
     add_matmul_rec(A, B, C, m, n, p, n, p, p);
}
```

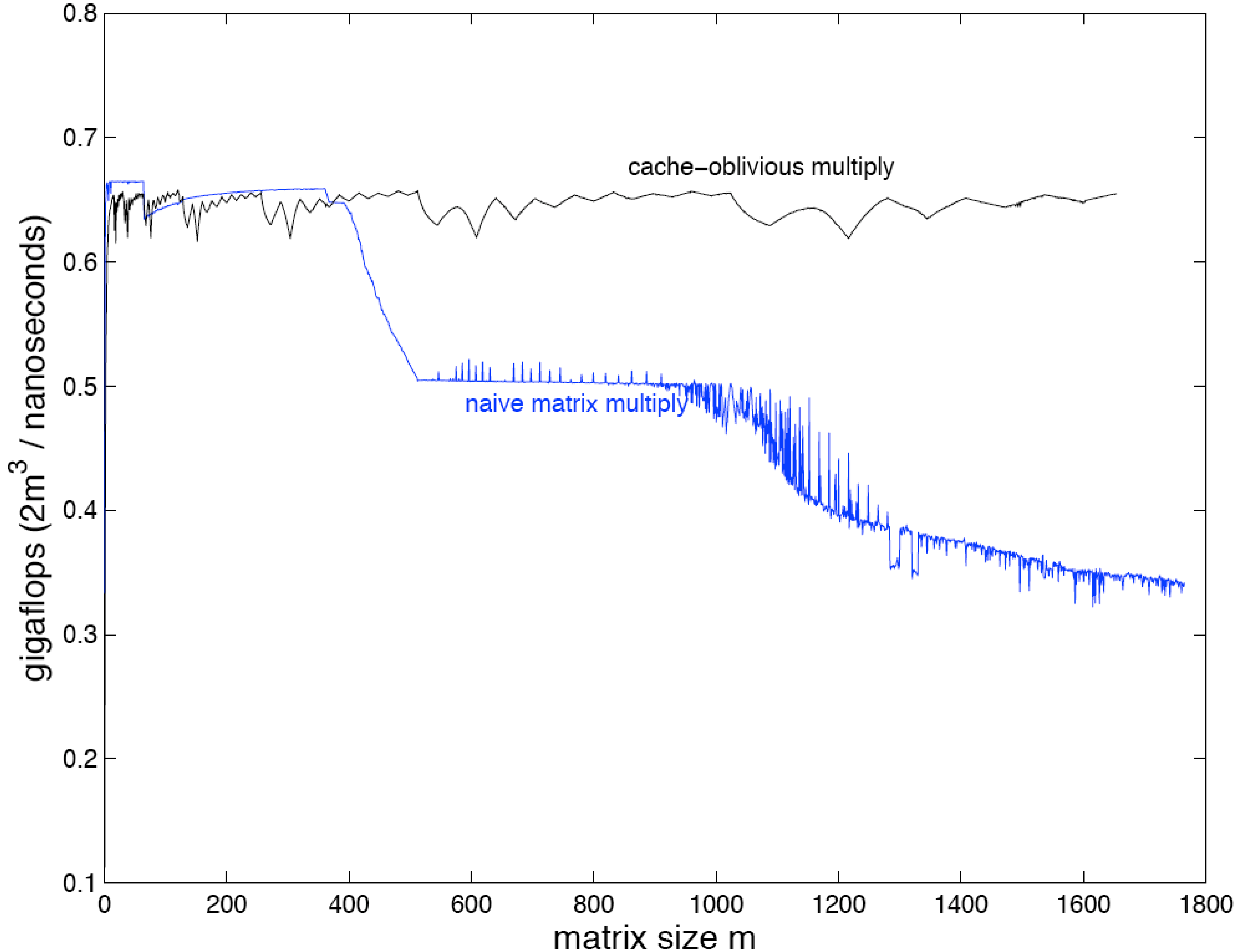
note: base case is ~16×16

recursing down to 1×1 would kill performance

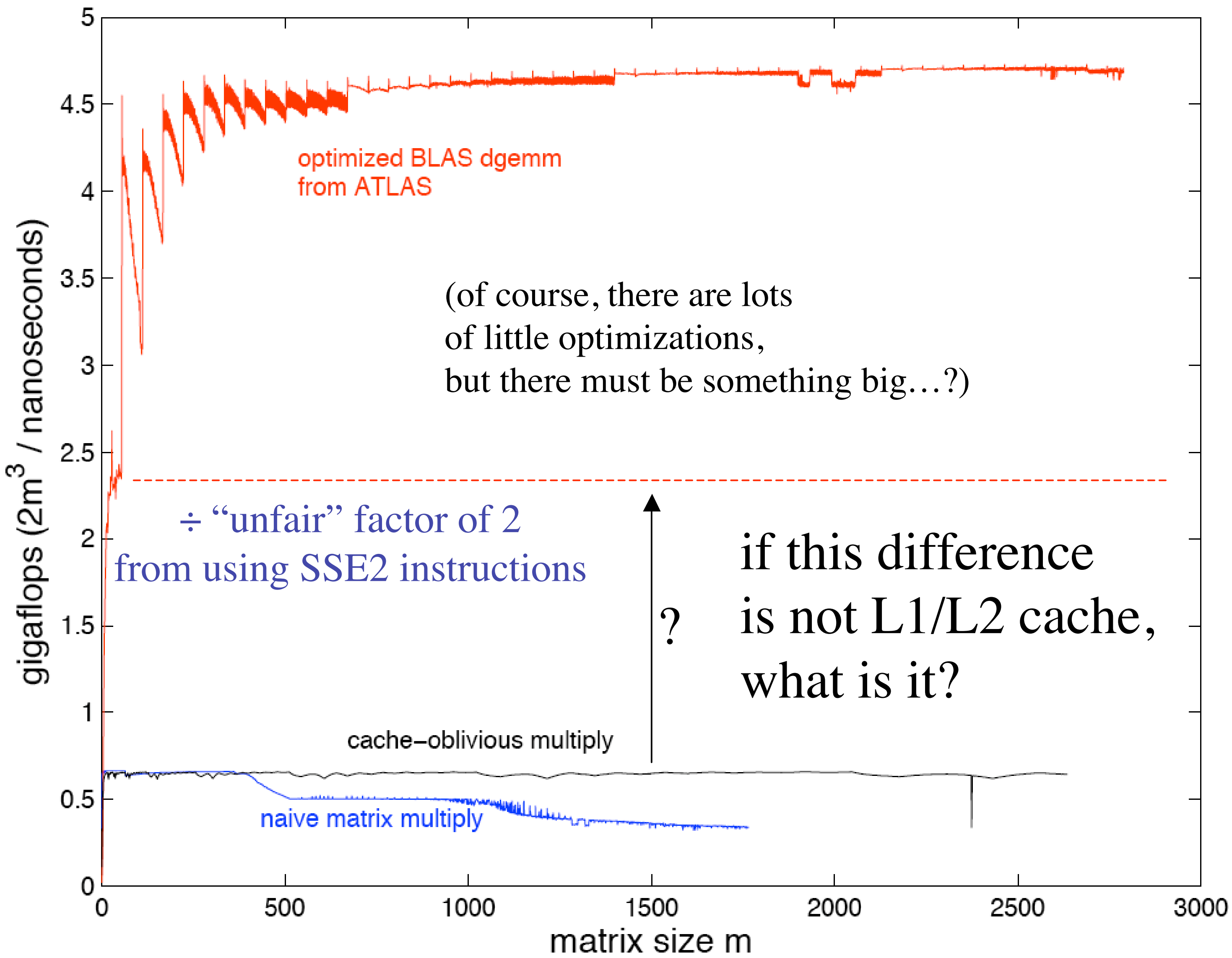
(1 function call per element, no register re-use)

dividing *C* into 4 — note that, instead, for very non-square matrices, we might want to divide *C* in 2 along longest axis

No Cache-based Performance Drops!



…but absolute performance still sucks



Registers .EQ. Cache

- The registers (~100) form a very small, almost ideal cache
  - Three nested loops is not the right way to use this "cache" for the same reason as with other caches
- Need long blocks of unrolled code: load blocks of matrix into local variables (= registers), do matrix multiply, write results
  - Loop-free blocks = many optimized hard-coded base cases of recursion for different-sized blocks … often automatically generated (ATLAS)
  - Unrolled $n \times n$ multiply has $(n^3)!$ possible code orderings — compiler cannot find optimal schedule (NP hard) — cache-oblivious scheduling can help (c.f. FFTW), but ultimately requires some experimentation (automated in ATLAS)