Pak

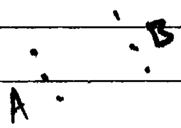
11/28/05

Last time: $A_i = (x_i, y_i)$ $\quad B_i = (u_i, v_i)$ $\quad i = 1 \dots n$

$\Sigma \stackrel{!}{=}$ set of non-intersecting $A \to B$ paths

e.g. all must go up + to the right $\leftarrow$ Assume all paths in Σ are $A_i \to B_i$ $\quad i = 1 \dots n$

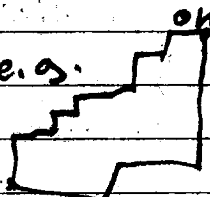
Then $|\Sigma| = \det\left(\left(\binom{u_i - x_i + v_j - y_j}{u_i - x_i}\right)\right)_{1 \le i,j \le n}$



Compute # of $\diamond$ polyminoes of perimeter $2n$

$\diamond$ polyminoes means polymino s.t. all vertical + horizontal lines intersect w/ one interval, + $\exists$ lower left + upper right.

e.g.



Note that once you pick the lower + upper corners $(0,0)$ + $(k, n-k)$, it's just # non-intersecting $(1,0) \to (k-1, n-k)$, $(0,1) \to (k, n-k-1)$ paths $= \alpha_k$, so # $\diamond$ poly$(n) = \sum_{k=1}^{n-1} \alpha_{k,n}$

$$\alpha_{k,n} = \det\begin{pmatrix} \binom{n-2}{k-1} & \binom{n-2}{k} \\ \binom{n-2}{k-2} & \binom{n-2}{k-1} \end{pmatrix}$$

$$\frac{(n-2)!^2}{(k-1)!\,(n-k)!\,k!\,(n-k-1)!} \det\begin{pmatrix} n-k & n-k-1 \\ k-1 & k \end{pmatrix}$$

$ab - (a-1)(b-1)$

$k(n-k) - (n-k-1)(k-1) = n-1$

$$\Rightarrow \det = \frac{(n-2)!\,(n-1)!}{(k-1)!\,k!\,(n-k)!\,(n-k-1)!} = \binom{n-1}{k}\binom{n-1}{k-1}\frac{1}{n-1}.$$

Narayana #s $N(i,n) = \frac{1}{n}\binom{n}{i}\binom{n}{i-1}$

Claim: $\sum_{i=1}^{n} N(i,n) = Cat(n)$

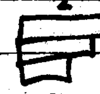
Pf: $Cat(n) =$ # binary trees w/ n vertices

$N(i,n) =$ # binary trees w/ n vertices + $i-1$ left edges

(pf by induction)

Solid partitions

Usual partition is $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell$

Can draw a partition  λ_1 λ_2 λ_3 Thus $\binom{a+b}{a}$ partitions fit a×b rectangle

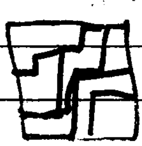
Now consider a×b×c box in $\mathbb{R}^3$

$P \subset [a] \times [b] \times [c]$ is a solid partition if

$\forall (x,y,z) \in P$ $(x-1,y,z), (x,y-1,z), (x,y,z-1) \in P$ (if defined, i.e. ≥ 0)

Thm (P. MacMahon)

solid partitions which fit a×b×c $M(a,b,c) = \prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$



etc.

Pf: Look at c-levels (slices at c=1, 2, ...)
This gives family of a×b partitions containing each other. Stretch it out by translating each i^{th} one up + over i steps. Now have non intersecting paths as outlines of partition levels

So thm ⇒ $M(a,b,c) = \det\begin{pmatrix} \binom{a+b}{a} & \binom{a+b}{a+1} & \binom{a+b}{a+2} \\ & \binom{a+b}{a} & \ddots \\ & & \ddots \end{pmatrix}$

this cancels out to WWTS.

(This is all in EC2)