## Exercises 6

(1) Let $\mathcal{A}$ be a central arrangement in $\mathbb{R}^n$ with distance enumerator $D_{\mathcal{A}}(t)$ (with respect to some base region $R_0$). Define a graph $G_{\mathcal{A}}$ on the vertex set $\mathcal{R}(\mathcal{A})$ by putting an edge between $R$ and $R'$ if $\#\mathrm{sep}(R, R') = 1$ (i.e., $R$ and $R'$ are separated by a unique hyperplane).
  (a) [2–] Show that $G_{\mathcal{A}}$ is a bipartite graph.
  (b) [2] Show that if $\#\mathcal{A}$ is odd, then $D_{\mathcal{A}}(-1) = 0$.
  (c) [2] Show that if $\#\mathcal{A}$ is even and $r(\mathcal{A}) \equiv 2 \pmod 4$, then $D_{\mathcal{A}}(-1) \equiv 2 \pmod 4$ (so $D_{\mathcal{A}}(-1) \neq 0$).
  (d) [2] Give an example of (c), i.e., find $\mathcal{A}$ so that $\#\mathcal{A}$ is even and $r(\mathcal{A}) \equiv 2 \pmod 4$.
  (e) [2] Show that (c) cannot hold if $\mathcal{A}$ is supersolvable. (It is *not* assumed that the base region $R_0$ is canonical. Try to avoid the use of Section 1.6.4.)
  (f) [2+] Show that if $\#\mathcal{A}$ is even and $r(\mathcal{A}) \equiv 0 \pmod 4$, then it is possible for $D_{\mathcal{A}}(-1) = 0$ and for $D_{\mathcal{A}}(-1) \neq 0$. Can examples be found for $\mathrm{rank}(\mathcal{A}) \leq 3$?

(2) [2–] Show that a sequence $(c_1, \ldots, c_n) \in \mathbb{N}^n$ is the inversion sequence of a permutation $w \in \mathfrak{S}_n$ if and only if $c_i \leq i - 1$ for $1 \leq i \leq n$.

(3) [2] Show that all cars can park under the scenario following Definition 1.1 if and only if the sequence $(a_1, \ldots, a_n)$ of preferred parking spaces is a parking function.

(4) [5] Find a bijective proof of Theorem 1.2, i.e., find a bijection $\varphi$ between the set of all rooted forests on $[n]$ and the set $\mathrm{PF}_n$ of all parking functions of length $n$ satisfying $\mathrm{inv}(F) = \binom{n+1}{2} - a_1 - \cdots - a_n$ when $\varphi(F) = (a_1, \ldots, a_n)$. NOTE. In principle a bijection $\varphi$ can be obtained by carefully analyzing the proof of Theorem 1.2. However, this bijection will be of a messy recursive nature. A "nonrecursive" bijection would be greatly preferred.

(5) [5] There is a natural two-variable refinement of the distance enumerator (9) of $\mathcal{S}_n$. Given $R \in \mathcal{R}(\mathcal{S}_n)$, define $d_0(R_0, R)$ to be the number of hyperplanes $x_i = x_j$ separating $R_0$ from $R$, and $d_1(R_0, R)$ to be the number of hyperplanes $x_i = x_j + 1$ separating $R_0$ from $R$. (Here $R_0$ is given by (7) as usual.) Set

$$D_n(q, t) = \sum_{R \in \mathcal{R}(\mathcal{S}_n)} q^{d_0(R_0,R)} t^{d_1(R_0,R)}.$$

What can be said about the polynomial $D_n(q, t)$? Can its coefficients be interpreted in a simple way in terms of tree or forest inversions? Are there formulas or recurrences for $D_n(q, t)$ generalizing Theorem 1.1, Corollary 1.1, or equation (5)? The table below give the coefficients of $q^i t^j$ in $D_n(q, t)$ for $2 \leq n \leq 4$.

| $t\backslash q$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | |

| $t\backslash q$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 1 |
| 1 | 2 | 2 | 2 | |
| 2 | 2 | 2 | | |
| 3 | 1 | | | |

| $t\backslash q$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 3 | 3 | 1 |
| 1 | 3 | 3 | 6 | 7 | 6 | 3 | |
| 2 | 5 | 5 | 8 | 9 | 5 | | |
| 3 | 6 | 7 | 9 | 6 | | | |
| 4 | 5 | 6 | 5 | | | | |
| 5 | 3 | 3 | | | | | |
| 6 | 1 | | | | | | |

Some entries of these table are easy to understand, e.g., the first and last entries in each row and column, but a simple way to compute the entire table is not known.

(6) [5–] Let $\mathcal{G}_n$ denote the generic braid arrangement

$$x_i - x_j = a_{ij}, \quad 1 \leq i < j \leq n,$$

in $\mathbb{R}^n$. Can anything interesting be said about the distance enumerator $D_{\mathcal{G}_n}(t)$ (which depends on the choice of base region $R_0$ and possibly on the $a_{ij}$'s)? Generalize if possible to generic graphical arrangments, especially for supersolvable (or chordal) graphs.

(7) [3–] Let $\mathcal{A}$ be a real supersolvable arrangement and $R_0$ a canonical region of $\mathcal{A}$. Show that the weak order $W_{\mathcal{A}}$ (with respect to $R_0$) is a lattice.

(8) (a) [2+] let $\mathcal{A}$ be a real central arrangement of rank $d$. Suppose that the weak order $W_{\mathcal{A}}$ (with respect to some region $R_0 \in \mathcal{R}(\mathcal{A})$) is a lattice. Show that $R_0$ is simplicial, i.e., bounded by exactly $d$ hyperplanes.

(b) [3–] Let $\mathcal{A}$ be a real central arrangement. Show that if every region $R \in \mathcal{R}(\mathcal{A})$ is simplicial, then $W_{\mathcal{A}}$ is a lattice.