Exercises 2

(1) [3–] Show that for any arrangement $\mathcal{A}$, we have $\chi_{c\mathcal{A}}(t) = (t-1)\chi_{\mathcal{A}}(t)$, where $c\mathcal{A}$ denotes the cone over $\mathcal{A}$. (Use Whitney's theorem.)

(2) [2–] Let G be a graph on the vertex set $[n]$. Show that the bond lattice L_G is a sub-join-semilattice of the partition lattice Π_n but is not in general a sublattice of Π_n.

(3) [2–] Let G be a forest (graph with no cycles) on the vertex set $[n]$. Show that $L_G \cong B_{E(G)}$, the boolean algebra of all subsets of $E(G)$.

(4) [2] Let G be a graph with n vertices and $\mathcal{A}_G$ the corresponding graphical arrangement. Suppose that G has a k-element clique, i.e., k vertices such that any two are adjacent. Show that $k!|r(\mathcal{A})$.

(5) [2+] Let G be a graph on the vertex set $[n] = \{1, 2, \dots, n\}$, and let $\mathcal{A}_G$ be the corresponding graphical arrangement (over any field K, but you may assume $K = \mathbb{R}$ if you wish). Let $\mathcal{C}_n$ be the coordinate hyperplane arrangement, consisting of the hyperplanes $x_i = 0$, $1 \le i \le n$. Express $\chi_{\mathcal{A}_G \cup \mathcal{C}_n}(t)$ in terms of $\chi_{\mathcal{A}_G}(t)$.

(6) [4] Let G be a planar graph, i.e., G can be drawn in the plane without crossing edges. Show that $\chi_{\mathcal{A}_G}(4) \neq 0$.

(7) [2+] Let G be a graph with n vertices. Show directly from the the deletion-contraction recurrence (20) that

$$(-1)^n \chi_G(-1) = \#\mathrm{AO}(G).$$

(8) [2+] Let $\chi_G(t) = t^n - c_{n-1}t^{n-1} + \cdots + (-1)^{n-1}c_1 t$ be the chromatic polynomial of the graph G. Let i be a vertex of G. Show that c_1 is equal to the number of acyclic orientations of G whose unique source is i. (A *source* is a vertex with no arrows pointing in. In particular, an isolated vertex is a source.)

(9) [5] Let $\mathcal{A}$ be an arrangement with characteristic polynomial $\chi_{\mathcal{A}}(t) = t^n - c_{n-1}t^{n-1} + c_{n-2}t^{n-2} - \cdots + (-1)^n c_0$. Show that the sequence $c_0, c_1, \dots, c_n = 1$ is *unimodal*, i.e., for some j we have

$$c_0 \le c_1 \le \cdots \le c_j \ge c_{j+1} \ge \cdots \ge c_n.$$

(10) [2+] Let $f(n)$ be the total number of faces of the braid arrangement $\mathcal{B}_n$. Find a simple formula for the generating function

$$\sum_{n\ge 0} f(n)\frac{x^n}{n!} = 1 + x + 3\frac{x^2}{2!} + 13\frac{x^3}{3!} + 75\frac{x^4}{4!} + 541\frac{x^5}{5!} + 4683\frac{x^6}{6!} + \cdots.$$

More generally, let $f_k(n)$ denote the number of k-dimensional faces of $\mathcal{B}_n$. For instance, $f_1(n) = 1$ (for $n \ge 1$) and $f_n(n) = n!$. Find a simple formula for the generating function

$$\sum_{n\ge 0}\sum_{k\ge 0} f_k(n) y^k \frac{x^n}{n!} = 1 + yx + (y + 2y^2)\frac{x^2}{2!} + (y + 6y^2 + 6y^3)\frac{x^3}{3!} + \cdots.$$