18.311 — MIT (Spring 2014)

Rodolfo R. Rosales

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Problem Set # 03. Due: Fri. March 14.

IMPORTANT:

— Turn in the regular and the special problems **stapled in two SEPARATE** packages.
— **Print your name** in each page of your answers.
— In page one of each package **print the names** of the other members of your group.

Update: fixed a problem with problem 1.4 below.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 61.03.

Assume that a velocity field, $u = u(x,t)$ exists. Show that the acceleration of an individual car is given by

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}. \tag{1.1}$$

1.2 Statement: Haberman problem 63.06.

Assume that $u = u(\rho)$. If α denotes the car's acceleration, show that

$$\alpha = -\rho\,\frac{du}{d\rho}\,\frac{\partial u}{\partial x}. \tag{1.2}$$

Is the minus sign here reasonable?

1.3 Statement: Haberman problem 67.01.

Suppose that

$$u = u_{\mathsf{max}}\left(1 - \frac{\rho}{\rho_{\mathsf{max}}}\right). \tag{1.3}$$

(a) What is the density wave velocity if the traffic density is nearly ρ_0 everywhere.

(b) Show that this density wave velocity is less than the car's velocity.

1.4 Statement: Haberman problem 71.02.

Consider the flow-density relationship used by H. Greenberg[1] to match Lincoln Tunnel data

$$q = a\,\rho\,\ln\left(\frac{\rho_{\mathsf{max}}}{\rho}\right),$$

where $a > 0$ is a constant. **Show that** the density wave velocity relative to a moving car is the same constant no matter what the density.

[1] Greenberg H., *An Analysis of Traffic Flow*, Operations Research, **7**:79–85, (1959).

1.5 Problem 6302a. State laws and road capacity.

If cars obey (exactly) the typical state law[2] on following distances: (1) What is the road capacity? (2) At what density and velocity does this maximum occur? (3) Will increasing the speed limit increase the road's capacity? (4) Draw a plot of the flux q versus the density ρ. (5) Draw a plot of the flux q versus the velocity u.
The relationship of the car velocity u to the car density ρ prescribed by the state laws has the form

$$u = \mathsf{min}\left\{u_M,\, V\,\left(\frac{1}{\rho\, L} - 1\right)\right\}. \tag{1.4}$$

Here L is the car length, V is the "trigger" velocity in the law,[3] and u_M is the speed limit. Assume that $L = 16$ ft $= (1/330)$ mile, $V = 10$ m.p.h., and $u_M = 50$ m.p.h. (80 k.p.h.).

1.6 Statement: Linear 1st order PDE # 05.

Consider the problem

$$x\, u_x + (x+y)\, u_y = 1, \qquad \text{with} \qquad u(1,\, y) = y \;\text{ for }\; 0 < y < 1. \tag{1.5}$$

Question 1. Write the solution $u = u(x,\, y)$ in the region where it is uniquely determined.
Question 2. Describe the region in the plane where the solution to (1.5) is uniquely determined.
Question 3. Write all the functions $u = u(x,\, y)$ that satisfy (1.5) on $x > 0$ and $-\infty < y < \infty$.
Question 4. Write all the functions $u = u(x,\, y)$ that satisfy the pde in (1.5) for $x < 0$.
Question 5 *(challenge/optional).* What happens along $x = 0$? Can you produce solutions to the pde that are continuous in the "punctured" plane (plane minus the origin)?

2 Special Problems.

2.1 Statement: TFPa14. Solve using dimensionless variables.

Consider the traffic flow equation

$$\boldsymbol{\rho_t + q_x = 0,} \tag{2.6}$$

with the quadratic flow $\boldsymbol{q = \frac{4\, q_m}{\rho_j^2}\, \rho\, (\rho_j - \rho)}$, where q_m = road capacity and ρ_j = jamming density.
Assume that either

[2] See equation (1.4).
[3] As in: Keep the distance to the next car at one car length L per every V in speed.

(a) $\boldsymbol{\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } x \le 0 \\ \dfrac{\rho_0}{1 + (x/d)^2} & \text{for } x \ge 0 \end{cases}}$ or **(b)** $\boldsymbol{\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } x \le 0 \\ \dfrac{\rho_0}{1 + (x/d)} & \text{for } x \ge 0 \end{cases}}$

where $\boldsymbol{0 < \rho_0 < \rho_j}$ and $d > 0$ are constants. Then

1. **Write everything using dimensionless variables.**
2. **Write the solution to (a) and (b) above using characteristics.**
3. **Show that there are no characteristic crossings. Thus (2) gives the complete solution.**
4. **In the case of (b), solve for $\boldsymbol{\xi = \xi(x,\,t)}$, and write the solution explicitly.**

Note: in **(4)** ξ denotes the label for the characteristics, defined as follows: $x = \xi$ at $t = 0$.

2.2 Statement: TFPa19. Linear first order equation and characteristics.

Find the solution to

$$x\,\psi_x - y\,\psi_y + \psi = 2\,x\,, \quad \text{with } \psi = 0 \text{ on the curve } \Gamma \text{ given by } y = 1. \tag{2.7}$$

Draw the characteristics which touch Γ. Where is the solution defined?

THE END.