

Solutions to Homework 3

$$\textcircled{1} \quad \int_0^1 dy \, K(x,y) u(y) = 1, \quad \text{where} \quad K(x,y) = \begin{cases} xy + (x-y)^{-1/2}, & x > y. \\ xy, & x < y. \end{cases}$$

$$1 = \int_0^1 dy \, K(x,y) u(y) = \int_0^x dy \, [xy + (x-y)^{-1/2}] u(y) + \int_x^1 dy \, xy u(y)$$

$$= \int_0^1 dy \, xy u(y) + \int_0^x dy \, (x-y)^{-1/2} u(y)$$

$$\Rightarrow \int_0^1 dy \, \frac{u(y)}{(x-y)^{1/2}} = 1 - x \int_0^1 dy \, y u(y).$$

$$\text{Let} \quad \alpha \equiv \int_0^1 dy \, y u(y) \tag{1}$$

$$\text{The integral equation reads} \quad \int_0^x dy \, \frac{u(y)}{\sqrt{x-y}} = 1 - \alpha x. \tag{2}$$

This is Abel's integral equation, i.e., a Volterra equation with the translationally invariant kernel  $(x-y)^{-1/2}$ , by treating  $\alpha$  as given. Apply Laplace transform.

With  $K(x,y) = k(x-y) = (x-y)^{-1/2}$  and  $f(x) = 1 - \alpha x$ , the Laplace transforms

of  $k(x)$  and  $f(x)$  are

$$\bar{k}(s) = \int_0^\infty dx \, e^{-sx} x^{-1/2} = \frac{1}{\sqrt{s}} \int_0^\infty dt \, e^{-t} t^{-1/2} = \sqrt{\frac{\pi}{s}},$$

$$\bar{f}(s) = \int_0^\infty dx \, (1 - \alpha x) \cdot e^{-sx} = \frac{1}{s} - \frac{\alpha}{s^2}.$$

The application of Laplace transform to both sides of (2) gives

$$\bar{k}(s) \cdot \bar{u}(s) = \bar{f}(s) \Rightarrow \sqrt{\frac{\pi}{s}} \cdot \bar{u}(s) = \frac{1}{s} - \frac{\alpha}{s^2} \quad \therefore \bar{u}(s) = \frac{1}{\sqrt{\pi s}} \left(1 - \frac{\alpha}{s}\right)$$

Inversion of this expression furnishes

$$u(x) = \frac{1}{\pi} \frac{1}{\sqrt{x}} (1 - 2\alpha x) \quad (\text{why?})$$

It remains to determine  $\alpha$ . By virtue of (1),

$$\begin{aligned} \alpha &= \int_0^1 dy \cdot y u(y) = \int_0^1 dy \cdot y \cdot \frac{1}{\pi} \frac{1}{\sqrt{y}} (1 - 2\alpha y) = \frac{1}{\pi} \left[ \frac{2}{3} y^{3/2} \Big|_0^1 - 2\alpha \frac{2}{5} y^{5/2} \Big|_0^1 \right] \\ &= \frac{1}{\pi} \left( \frac{2}{3} - \frac{4\alpha}{5} \right) \quad \Rightarrow \quad \alpha = \frac{2/3\pi}{1 + 4/5\pi} = \frac{10}{15\pi + 12} \end{aligned}$$

Hence,

$$u(x) = \frac{1}{\pi} \frac{1}{\sqrt{x}} \left( 1 - \frac{20}{15\pi + 12} x \right).$$

$$\textcircled{3} \quad u(x) = 1 + \lambda \int_x^{\infty} dy e^{a(x-y)} u(y), \quad a > 0, \quad -\infty < x < \infty.$$

(a) The kernel of this equation is  $K(x,y) = e^{a(x-y)}$ , with the norm squared

$$\|K\|^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |K(x,y)|^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{2a(x-y)} = \int_{-\infty}^{\infty} dx e^{2ax} \int_{-\infty}^{\infty} dy e^{-2ay} = +\infty.$$

Hence, the kernel is not square integrable. (We should not, therefore, expect for sure that the eigenvalues of the homogeneous equation form a discrete set.)

$$(b) \text{ Homogeneous equation: } u(x) = \lambda \int_x^{\infty} dy e^{a(x-y)} u(y), \quad -\infty < x < \infty, \quad a > 0.$$

We solve this equation by converting it into a differential equation. But before we do so, we notice that a requirement on the solution is that the integral on the RHS of this equation should converge:

$$\int_x^{\infty} dy e^{a(x-y)} u(y) < \infty$$

for any finite  $x$ .

Differentiation of both sides of this integral equation gives

$$u'(x) = -\lambda u(x) + a \underbrace{\lambda \int_x^{\infty} dy e^{a(x-y)} u(y)}_{u(x)} \Leftrightarrow u'(x) + (\lambda - a) u(x) = 0.$$

It follows that  $u(x) = C_1 e^{-(\lambda - a)x}$ ,  $C_1$ : arbitrary constant.

Check convergence of integral:

$$\int_x^{\infty} dy e^{a(x-y)} u(y) = C_1 \int_x^{\infty} dy e^{a(x-y)} e^{-(\lambda - a)y} = C_1 e^{ax} \int_x^{\infty} dy e^{-\lambda y} < \infty \Rightarrow \boxed{\lambda > 0}$$

Hence, the spectrum  $\mathcal{S}$  is continuous:  $\mathcal{S} = \{ \lambda : \lambda > 0 \}$

$$(c) \quad u(x) = 1 + \lambda \int_x^\infty dy e^{a(x-y)} u(y), \quad a > 0, \quad -\infty < x < \infty$$

Differentiation in  $x$  gives:  $u'(x) = -\lambda u(x) + a \underbrace{\lambda \int_x^\infty dy e^{a(x-y)} u(y)}_{u(x)-1}$

$$\Rightarrow u'(x) + (\lambda - a)u(x) = -a.$$

Solution:  $u(x) = K e^{-(\lambda-a)x} - \frac{a}{\lambda-a} = K' e^{-(\lambda-a)x} - a \frac{1 - e^{-(\lambda-a)x}}{\lambda-a}$   
 by setting  $K = K' + a$  so as to avoid the ambiguity at  $\lambda = a$ ;  $K$  and  $K'$  are constants.

[Indeed, for  $\lambda = a$ , the RHS gives  $u(x) \xrightarrow{\lambda \rightarrow a} K' - a \lim_{\lambda \rightarrow a} \frac{1 - e^{-(\lambda-a)x}}{\lambda-a}$   
 $= K' - a \lim_{\lambda \rightarrow a} \frac{\frac{d}{d\lambda} [1 - e^{-(\lambda-a)x}]}{\frac{d}{d\lambda} (\lambda-a)} = K' - a \frac{x}{1} = K' - ax$ , in agreement with

solving  $u'(x) = -a$ . ]

Again, the integral of the original equation has to converge:

$$\begin{aligned} \int_x^\infty dy e^{a(x-y)} u(y) &= \int_x^\infty dy e^{a(x-y)} \left[ K e^{-(\lambda-a)y} - \frac{a}{\lambda-a} \right] \\ &= K e^{ax} \int_x^\infty dy e^{-\lambda y} - \frac{a}{\lambda-a} e^{ax} \frac{1}{a} e^{-ax} = K e^{ax} \int_x^\infty dy e^{-\lambda y} - \frac{1}{\lambda-a} < \infty. \end{aligned}$$

It follows that for  $\boxed{\lambda > 0}$ ,  $K$  can be an arbitrary <sup>constant</sup> and the solution has one arbitrary constant:

$$u(x) = K e^{-(\lambda-a)x} - \frac{a}{\lambda-a}, \quad \underline{\lambda > 0}.$$

(As we said before,  $\lambda = a$  is not a singularity since we can get a finite solution by "renormalizing" or "redefining"  $K$ .)

For  $\boxed{\lambda \leq 0}$ , the only way to have a convergent integral is to set  $\boxed{K = 0}$ :

$$u(x) = -\frac{a}{\lambda-a}, \quad \underline{\lambda \leq 0}.$$

Our results can be summarized as follows, by denoting the solution of the homogeneous equation as  $u_h(x) = K e^{-(\lambda-a)x}$  and the spectrum as  $\mathcal{S} = \{\lambda: \lambda > 0\}$ :

$$u(x) = \begin{cases} -\frac{a}{\lambda-a} + u_h(x), & \lambda \in \mathcal{S} \quad (\lambda \text{ belongs to } \mathcal{S}) \\ -\frac{a}{\lambda-a}, & \lambda \notin \mathcal{S}. \quad (\lambda \text{ does NOT belong to } \mathcal{S}). \end{cases}$$

(Alternatively,  $u(x) = -\frac{a}{\lambda-a} + u_h(x)$  for all  $x$ , where  $u_h(x) = \begin{cases} 0, & \lambda \notin \mathcal{S} \\ K e^{-(\lambda-a)x}, & \lambda \in \mathcal{S} \end{cases}$ .)

③ Let  $\phi(x) = \int_{-\pi}^{\pi} dy e^{k \cos x \cos y} \psi(y)$ , where  $\psi(y)$  satisfies the equation

$$\frac{d^2 \psi}{dy^2} + (a^2 + k^2 \cos^2 y) \psi(y) = 0; \text{ further, } \psi(y) \text{ is even and periodic.}$$

Clearly,  $\phi(-x) = \phi(x)$  and  $\phi(x+2\pi) = \phi(x)$ , i.e.,  $\phi(x)$  is even and periodic.

We will show that  $\phi(x)$  satisfies  $\frac{d^2 \phi}{dx^2} + (a^2 + k^2 \cos^2 x) \phi(x) = 0$

$$\frac{d^2 \phi}{dx^2} + (a^2 + k^2 \cos^2 x) \phi(x) = \int_{-\pi}^{\pi} dy \left[ \frac{\partial^2}{\partial x^2} e^{k \cos x \cos y} + (a^2 + k^2 \cos^2 x) e^{k \cos x \cos y} \right] \psi(y)$$

$$= \int_{-\pi}^{\pi} dy \left( k^2 \overbrace{\sin^2 x}^{1-\cos^2 x} \cos^2 y - k \cos x \cos y + a^2 + k^2 \cos^2 x \right) e^{k \cos x \cos y} \psi(y)$$

$$= \int_{-\pi}^{\pi} dy \left[ k^2 \cos^2 x (1 - \cos^2 y) - k \cos x \cos y + a^2 + k^2 \cos^2 y \right] e^{k \cos x \cos y} \psi(y)$$

$$= \int_{-\pi}^{\pi} dy \left[ k^2 \cos^2 x \sin^2 y - k \cos x \cos y + a^2 + k^2 \cos^2 y \right] e^{k \cos x \cos y} \psi(y),$$

where  $(k^2 \cos^2 x \sin^2 y - k \cos x \cos y) e^{k \cos x \cos y} = \frac{\partial^2}{\partial y^2} e^{k \cos x \cos y}$ . Hence,

$$\begin{aligned} \frac{d^2 \phi}{dx^2} + (a^2 + k^2 \cos^2 x) \phi(x) &= \int_{-\pi}^{\pi} dy \left[ \frac{\partial^2}{\partial y^2} e^{k \cos x \cos y} + (a^2 + k^2 \cos^2 y) e^{k \cos x \cos y} \right] \psi(y) \\ &= \left[ \frac{\partial}{\partial y} e^{k \cos x \cos y} \right] \psi(y) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} dy \left[ \left( \frac{\partial}{\partial y} e^{k \cos x \cos y} \right) \psi'(y) - (a^2 + k^2 \cos^2 y) e^{k \cos x \cos y} \psi(y) \right] \\ &= - \left[ \frac{\partial}{\partial y} e^{k \cos x \cos y} \psi(y) \right]_{y=-\pi}^{\pi} + \int_{-\pi}^{\pi} dy e^{k \cos x \cos y} \left[ \frac{d^2 \psi(y)}{dy^2} + (a^2 + k^2 \cos^2 y) \psi(y) \right] \\ &= 0 \quad \text{if} \quad \frac{d^2 \psi}{dy^2} + (a^2 + k^2 \cos^2 y) \psi(y) = 0. \end{aligned}$$

It follows that  $\frac{d^2 \phi}{dx^2} + (a^2 + k^2 \cos^2 x) \phi(x) = 0$ .

Since  $\phi(x)$  is <sup>also</sup> even and periodic, it must differ by  $\psi(x)$  only by a constant, i.e.,

$$\psi(x) = \lambda \phi(x) \iff \psi(x) = \lambda \int_{-\pi}^{\pi} dy e^{k \cos x \cos y} \psi(y).$$

$$\textcircled{4} \quad \int_{-h}^h dy \quad K(x-y) \cdot I(y) = A \cdot \sin(k \cdot |x|) + C_1 \cdot \cos(kx)$$

For  $K(x) = K_{ap}(x)$  and, say,  $\int_{-h}^h dx \quad |I(x)| < \infty$ , the LHS of this equation is infinitely differentiable in  $x$  at  $x=0$ , i.e., it is a smooth function of  $x$  in  $(-h, h)$ . In contrast, the RHS is not continuously differentiable at  $x=0$  because of the  $\sin(k \cdot |x|)$  term that contains  $|x|$ . We are thus forced to conclude that there is no solution to this equation, at least no reasonable solution that is integrable (and, hence, physically admissible).

The application of Laplace transform to both sides of (2) gives

$$\bar{k}(s) \cdot \bar{u}(s) = \bar{f}(s) \Rightarrow \sqrt{\frac{\pi}{s}} \cdot \bar{u}(s) = \frac{1}{s} - \frac{\alpha}{s^2} \quad \therefore \bar{u}(s) = \frac{1}{\sqrt{\pi s}} \left(1 - \frac{\alpha}{s}\right)$$

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$$= \frac{1}{\pi} \left( \frac{2}{3} - \frac{4\alpha}{5} \right) \quad \Rightarrow \quad \alpha = \frac{2/3\pi}{1 + 4/5\pi} = \frac{10}{15\pi + 12}$$

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