18.307: Integral Equations — M.I.T. Department of Mathematics, Spring 2006

Solutions to Homework 2

④ (a)
$$\begin{cases} i\Psi_t(x,t) + \Psi_{xx}(x,t) = \overbrace{V(x,t)\,\Psi(x,t)}^{\rho(x,t;\Psi)}, & -\infty<x<\infty,\ t>0 \\ \Psi(x,0) = a(x): \text{given}. \end{cases}$$

Write
$$\Psi(x,t) = \Psi_0(x,t) + \Psi_p(x,t), \quad \text{where} \quad \underline{\Psi(x,t<0)\equiv 0}, \text{ and}$$

- $\Psi_0(x,t)$: solution to homogeneous equation $(\rho\equiv 0)$,
$$i\Psi_{0,t}(x,t) + \Psi_{0,xx}(x,t) = 0, \qquad t>0. \tag{1}$$

- $\Psi_p(x,t)$: particular solution to given PDE,
$$\Psi_p(x,t) = \int_0^\infty dt' \int_{-\infty}^\infty dx' \; G(x,t;x',t')\cdot \underbrace{V(x',t')\cdot\Psi(x',t')}_{\rho(x',t';\Psi)} \tag{2}$$
(lower limit must be zero because $\Psi(x,t)=0$ for $t<0$).

The Green function $G(x,t;x',t')$ satisfies
$$iG_t + G_{xx} = \delta(x-x')\cdot\delta(t-t'), \qquad -\infty < x,t < \infty. \tag{3}$$

We <u>choose</u> $\Psi_0(x,t)$ to satisfy the <u>given</u> initial condition, i.e.,
$$\Psi_0(x,t=0) = a(x). \tag{4}$$

Consider the Fourier integral for $\Psi_0(x,t)$:
$$\Psi_0(x,t) = \int_{-\infty}^\infty \frac{dq}{2\pi}\, e^{iqx}\,\tilde{\Psi}_0(q,t), \qquad t>0. \tag{5}$$

Then from (1) one gets
$$i\tilde{\Psi}_{0,t}(q,t) - q^2\,\tilde{\Psi}_0(q,t) = 0. \tag{6}$$

It follows that
$$\tilde{\Psi}_0(q,t) = G(q)\cdot e^{-iq^2t}$$
$$\Rightarrow \quad \Psi_0(x,t) = \int_{-\infty}^{\infty}\frac{dq}{2\pi}\; G(q)\cdot e^{iqx-iq^2t} .$$

We apply the initial condition (4) in order to determine $G(q)$:
$$a(x) = \int_{-\infty}^{\infty}\frac{dq}{2\pi}\; G(q)\cdot e^{iqx} \qquad \therefore \quad G(q) = \int_{-\infty}^{\infty} dx\; e^{-iqx}\; a(x) \;:\text{known},$$

assuming that the Fourier transform of $a(x)$ exists.

Hence,
$$\begin{aligned}
\Psi_0(x,t) &= \int_{-\infty}^{\infty}\frac{dq}{2\pi}\; e^{iqx-iq^2t}\cdot\int_{-\infty}^{\infty} dx'\; e^{-iqx'}\, a(x') \\
&= \int_{-\infty}^{\infty} dx'\; a(x')\int_{-\infty}^{\infty}\frac{dq}{2\pi}\; e^{iq(x-x')-iq^2t} \\
&= \int_{-\infty}^{\infty} dx'\; a(x')\int_{-\infty}^{\infty}\frac{dq}{2\pi}\; e^{-it\left(q-\frac{x-x'}{2t}\right)^2 + i\frac{(x-x')^2}{4t}} \\
&= \int_{-\infty}^{\infty} dx'\; a(x')\; e^{i\frac{(x-x')^2}{4t}}\;\underbrace{\int_{-\infty}^{\infty}\frac{dq}{2\pi}\; e^{-itq^2}}_{I}\;, \qquad t>0.
\end{aligned}$$

The inner integral is evaluated by deforming the path of integration in the q-plane: we rotate the path by $-\pi/4$ so that $q = y\cdot e^{-i\pi/4}$:
$$I = \int_{-\infty}^{\infty}\frac{d(y\cdot e^{-i\pi/4})}{2\pi}\; e^{-it(y\cdot e^{-i\pi/4})^2} = \frac{e^{-i\pi/4}}{2\pi}\int_{-\infty}^{\infty} dy\; e^{-ty^2} = \frac{e^{-i\pi/4}}{2\pi}\sqrt{\frac{\pi}{t}}. \qquad (7)$$

Thus,
$$\Psi_0(x,t) = \frac{e^{-i\pi/4}}{2\pi}\sqrt{\frac{\pi}{t}}\int_{-\infty}^{\infty} dx'\; a(x')\; e^{i\frac{(x-x')^2}{4t}} . \qquad (8)$$

We proceed to calculate the Green function $G(x,t;x',t')$. We write

$$G(x,t;x',t') = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \, e^{iq_1(x-x') - iq_0(t-t')} \, \tilde{G}(q_1, q_0)$$

(3) $\Rightarrow$ $$(q_0 - q_1^2)\, \tilde{G}(q_1,q_0) = 1 \quad \Rightarrow \quad \tilde{G}(q_1,q_0) = \frac{1}{q_0 - q_1^2} .$$

$$\therefore \quad G(x,t;x',t') = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \, \frac{e^{iq_1(x-x') - iq_0(t-t')}}{q_0 - q_1^2} . \qquad (9)$$

Given that $\Psi_0(x,t=0) = a(x) = \Psi(x,t=0)$, it follows from $\Psi = \Psi_0 + \Psi_P$ and (2) that G satisfies

$$G(x,t=0;x',t') = 0 . \qquad (10)$$

The only choice for the path in the q_0-plane, according to (9), that can accomodate (10) is taking the path <u>above</u> the pole of the integrand at $q_0 = q_1^2$. Consequently,

$$G(x,t;x',t') = 0 \qquad \text{for} \qquad \underline{t < t'} .$$

For $\underline{t > t'}$ we close the path by a large semicircle in the lower plane and pick up a residue at $q_0 = q_1^2$:

$$G(x,t;x',t') = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \, \frac{-2\pi i}{2\pi} \, \frac{e^{iq_1(x-x') - iq_1^2(t-t')}}{1} \qquad \text{(see (7))}$$

$$= -i \, e^{i\frac{(x-x')^2}{4(t-t')}} \, \frac{e^{-i\pi/4}}{2\pi} \sqrt{\frac{\pi}{t-t'}} . \qquad (11)$$

(b) The desired integral equation is

$$\Psi(x,t) = \underbrace{\frac{e^{-i\pi/4}}{2\pi}\sqrt{\frac{\pi}{t}}\int_{-\infty}^{\infty} dx'\, a(x')\, e^{i\frac{(x-x')^2}{4t}}}_{\Psi_0(x,t)} - \frac{e^{i\pi/4}}{2\pi}\int_0^t dt' \int_{-\infty}^{\infty} dx' \sqrt{\frac{\pi}{t-t'}}\, e^{i\frac{(x-x')^2}{4(t-t')}} \underbrace{V(x',t')\Psi(x',t')}_{\rho}$$

This equation can be solved approximately by iteration if V is "sufficiently" small. To zeroth order we take $V \simeq 0$ and find

$$\Psi(x,t) \simeq \Psi_0(x,t).$$

To the next order,

$$\Psi(x,t) \simeq \Psi_0(x,t) - \frac{e^{i\pi/4}}{2\pi}\int_0^t dt' \int_{-\infty}^{\infty} dx' \sqrt{\frac{\pi}{t-t'}}\, e^{i\frac{(x-x')^2}{4(t-t')}} V(x',t')\, \Psi_0(x',t')$$

etc.

⑤ $$\begin{cases} u_{tt} - u_{xx} = \overbrace{p(x,t) - \lambda u_{xx} u_x^2}^{\rho} , & -\infty < x < \infty , \; t > 0. \\ u(x,t=0) = a(x) , \quad u_t(x,t=0) = b(x) . \end{cases}$$

$u(x,t) = u_0(x,t) + u_p(x,t)$; u_0 satisfies PDE with $\rho \equiv 0$.

$$u_{0,tt} - u_{0,xx} = 0 , \qquad (1)$$

Particular soln: $$u_p(x,t) = \int_0^\infty dt' \int_{-\infty}^\infty dx' \; G(x,t;x',t') \left[p(x',t') - \lambda u_{xx}(x',t') \cdot u_x^2(x',t') \right] , \qquad (2)$$

where we set $u(x,t) = 0$ for $t < 0$ and assumed $p(x,t) = 0$ for $t < 0$, and

$$G_{tt} - G_{xx} = \delta(x - x') \cdot \delta(t - t'). \qquad (3)$$

We require that $u_0(x,t)$ satisfies the conditions for $u(x,t)$,

$$\begin{cases} u_0(x,t=0) = a(x) \\ u_{0,t}(x,t=0) = b(x) . \end{cases}$$

Let $$u_0(x,t) = \int_{-\infty}^\infty \frac{dq}{2\pi} \, e^{iqx} \, \tilde{u}_0(q,t) , \qquad \tilde{u}_{0,tt} + q^2 \tilde{u}_0 = 0 ,$$

$$\tilde{u}_0(q,t) = A(q) \cdot \cos(qt) + B(q) \cdot \sin(qt) ,$$

$$u_0(x,t) = \int_{-\infty}^\infty \frac{dq}{2\pi} \, e^{iqx} \left[A(q) \cos(qt) + B(q) \sin(qt) \right] .$$

$$\left. \begin{aligned} u_0(x,0) = a(x) &\rightarrow A(q) = \int_{-\infty}^\infty dx \; e^{-iqx} \, a(x) \\ u_{0,t}(x,0) = b(x) &\rightarrow q B(q) = \int_{-\infty}^\infty dx \; e^{-iqx} \, b(x). \end{aligned} \right\}$$

: $u_0(x,t)$ completely known, assuming $a(x)$, $b(x)$ satisfy usual integrability conditions.

Find $G(x,t;x',t')$ by applying F.T. to (3):

$$G(x,t;x',t') = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi}\, e^{iq_1(x-x') - iq_0(t-t')}\, \tilde{G}(q_1,q_0)\,,$$

$$(q_1^2 - q_0^2)\, \tilde{G}(q_1,q_0) = 1\,, \qquad \tilde{G}(q_1,q_0) = \frac{1}{q_1^2 - q_0^2}\,.$$

$$G(x,t;x',t') = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi}\, \frac{e^{iq_1(x-x') - iq_0(t-t')}}{q_1^2 - q_0^2}\,.$$

The path in q_0-plane is chosen under $\begin{cases} G(x,0;x',t'>0) = 0 \\ G_t(x,0;x',t'>0) = 0 \end{cases}$ since $u_p(x,0)=0$, $u_{p,t}(x,0)=0$.

It follows that $\underline{G(x,t;x',t') = 0 \text{ for } t<t'}$: <u>causal</u> Green's function,

and the path in the q_0-plane lies <u>above</u> the poles at $q_0 = \pm q_1$.

It follows that

$$G(x,t;x',t') = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi}\, e^{iq_1(x-x')}\, \frac{\sin[q_1(t-t')]}{q_1}\,, \qquad t>t'.$$

Following the calculation done in class,

$$G(x,t;x',t') = \begin{cases} \frac{1}{2}\,, & t-t' > |x-x'| \\ 0\,, & 0 < t-t' < |x-x'|. \end{cases}$$

Finally,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqx} \Big[\overbrace{A(q)\cos(qt) + B(q)\sin(qt)}^{\text{known}} \Big]$$
$$+ \frac{1}{2}\int_0^{t} dt' \int_{x-(t-t')}^{x+(t-t')} dx' \left[p(x',t') - \lambda\, u_{xx}(x',t') \cdot u_x^2(x',t') \right].$$

let

$$P(x,t) \equiv \int_0^{\infty} dt' \int_{-\infty}^{\infty} dx' \; G(x,t;x',t')\, p(x',t') = \frac{1}{2}\int_0^{t} dt' \int_{x-(t-t')}^{x+t-t'} dx' \; p(x',t')$$

Then,

$$u(x,t) = u_0(x,t) + P(x,t) - \frac{\lambda}{2}\int_0^{t} dt' \int_{x-(t-t')}^{x+(t-t')} dx' \; u_{xx}(x',t')\, u_x^2(x',t')$$

For $\lambda = 0 \rightarrow u(x,t) = u_0(x,t) + P(x,t)$, exactly.

For $\lambda \neq 0$, λ: small ,

$$u(x,t) \simeq u_0(x,t) + P(x,t) - \frac{\lambda}{2}\int_0^{t} dt' \int_{x-(t-t')}^{x+(t-t')} dx' \; \frac{\partial^2}{\partial x'^2}\left[u_0(x',t') + P(x',t')\right] \cdot \left(\frac{\partial}{\partial x'}[u_0 + P]\right)^2$$

replace u by $u_0 + P$ inside the integral.

So, for λ: small the equation is solved by iteration.

(6) $$f(x) = \int_0^x dy\; K(x-y)\, u(y), \qquad 0<x<\infty \;;\qquad K(x) = \ln x.$$

Apply the Laplace transform,
$$\bar{u}(s) = \int_0^\infty dx\; e^{-sx}\, u(x),$$
$$\bar{K}(s) = \int_0^\infty dx\; K(x)\, e^{-sx}, \qquad \bar{f}(s) = \int_0^\infty dx\; f(x)\, e^{-sx}:$$

$$\bar{f}(s) = \bar{K}(s)\cdot \bar{u}(s) \;\Rightarrow\; \bar{u}(s) = \frac{\bar{f}(s)}{\bar{K}(s)}.$$

It remains to find $\bar{K}(s)$:
$$\bar{K}(s) = \int_0^\infty dx\; e^{-sx} \ln x \overset{sx=t}{=\!=} \frac{1}{s}\int_0^\infty dt\; e^{-t} \ln(t/s)$$
$$= \frac{1}{s}\left[\int_0^\infty dt\; e^{-t}\ln t - \ln s \int_0^\infty dt\; e^{-t}\right] = \frac{1}{s}(-\gamma - \ln s) = -\frac{\gamma+\ln s}{s}.$$

Hence,
$$\bar{f}(s) = -\frac{\gamma+\ln s}{s}\,\bar{u}(s) \;\Longrightarrow\; \bar{u}(s) = -\frac{s}{\gamma+\ln s}\,\bar{f}(s)$$

Finally,
$$u(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\; e^{sx}\,\bar{u}(s) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\; e^{sx}\,\frac{s}{\gamma+\ln s}\,\bar{f}(s),$$

where c is a positive (real) number such that the path of integration lies to the <u>right</u> of <u>all</u> singularities of the integrand.