18.307: Integral Equations **M.I.T. Department of Mathematics**

Spring 2006

Homework 6 **Due:** Wednesday, 04/12/06

18. (Prob. 4.7 in text by M. Masujima.) Consider the equation $u(x) = \lambda \int_{-\infty}^{\infty} dy\, e^{-ixy}\, u(y),\ -\infty < x < \infty$, i.e., with a kernel that is not square integrable. Note that solutions to this equation are essentially "Fourier transforms of themselves."
(a) Show that there are only 4 eigenvalues λ of the kernel e^{-ixy}. What are they?
(b) Show by an explicit calculation that the functions $u_n(x) = e^{-x^2/2}\, H_n(x)$, where $H_n(x) = (-1)^n\, e^{x^2}\, \frac{d^n}{dx^n}\, e^{-x^2}$ are Hermite polynomials, are eigenfunctions with corresponding eigenvalues $(-i)^n/\sqrt{2\pi}$ $(n = 0,\, 1,\, 2,\, \ldots)$. Hence, the eigenvalues of part (a) are infinitely degenerate, i.e., there is a infinite number of independent eigenfunctions for each of them.
(c) Using the result in (b) and the fact that u_n are known to form a complete set, in some sense, show that any square integrable solution is of the form $u(x) = f(x) + \mathcal{C}\, \tilde{f}(x)$, where $f(x)$ is an (arbitrary) odd or even, square integrable function with Fourier transform $\tilde{f}(k)$, and $\mathcal{C}$ is a suitable constant. Evaluate $\mathcal{C}$ and relate its value(s) to the eigenvalues found in part (a).
(d) From (c), construct a solution to the original integral equation by taking $f(x) = e^{-ax^2/2}$ (Gaussian, $a > 0$).

19. (Prob. 5.11 in text by M. Masujima.) Consider the kernel K of a 2nd-kind Fredholm equation, which is given by

$$K(x, y) = \begin{cases} 3, & 0 \le y < x \le 1, \\ 2, & 0 \le x < y \le 1. \end{cases}$$

(a) Find the kernel eigenfunctions u_n and corresponding eigenvalues λ_n.
(b) Is K symmetric? Determine the transpose kernel K^T, and find its eigenfunctions v_n with corresponding eigenvalues λ_n.
(c) Show by an explicit calculation that any u_n is orthogonal to any v_m if $m \neq n$.
(d) Derive the spectral representation of $K(x, y)$ in terms of u_n and v_n

20. (Probs. 5.3 & 5.4, Chap. 6 in text by I. Stakgold.) The energy levels E of an atom that experiences *non-local* interactions with other atoms in a dilute gas are described by the eigenvalue problem $Au = E\, u$, where A is the *integrodifferential* operator defined by

$$Au = -\frac{d^2u}{dx^2} + \int_0^1 dy\, xy\, u(y), \quad 0 < x < 1,$$

and $u(x)$ is any function that has a second derivative continuous for $0 < x < 1$ and satisfies the boundary conditions $u(0) = 0$ and $u'(1) = 0$. We denote the space of such functions as D_A. **Beware**: The constant E above multiplies the u outside the integral.

 (a) Show that all eigenvalues E_n of this problem are real and positive, and that eigenfunctions corresponding to different eigenvalues are orthogonal. Justify your answer. Why is it sufficient to restrict ourselves to real eigenfunctions?

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(b) By using Green's function, show that the given problem can be reduced to a homogeneous integral equation (with no derivatives) involving a symmetric kernel $K(x, y)$, i.e., one needs to find the eigenvalues of a (pure) *integral* operator. **Hint:** Define $G(x, x')$ such that $-G_{xx} = \delta(x - x')$ by considering as 'source' $\rho(x') = Eu(x') - x' \int_0^1 dy\, yu$. What are the conditions on G? After you find G, calculate any undetermined constant in ρ for consistency.

(c) [Without using (b) above] By noticing that the given problem $Au = E\,u$ is of the form $u'' + Eu = bu$, show that the eigenvalues $E = \mu^2$ are obtained as positive roots of the equation

$$\tan\mu = \mu + \frac{\mu^3}{3} - \mu^5.$$

(d) Sketch the functions $\tan\mu$ and $\mu + \mu^3/3 - \mu^5$. Find an approximate value for the lowest eigenvalue, E_0, according to (c) above.

(e) According to a variational principle for the lowest eigenvalue E_0,

$$E_0 = \min_{v \in D_A} \frac{\int_0^1 dx\,[v'(x)]^2 + \left[\int_0^1 dx\, xv(x)\right]^2}{\int_0^1 dx\, v(x)^2}.$$

Can you explain this equation? Use the trial function $v(x) = x(c - x)$ to find an upper bound for E_0. What is c? Use the "trace inequality" for the iterated kernel K_2 of the kernel K of part (b) to find a lower bound to E_0. Compare your answer with the answer obtained in part (d).