3. Feynman calculus

3.1. **Wick's theorem.** Let V be a real vector space of dimension d with volume element dx. Let $S(x)$ be a smooth function on a box $\mathbf{B} \subset V$ which attains a minimum at $x = c \in \text{Interior}(B)$, and g be any smooth function on $\mathbf{B}$. In the last section we have shown that the function

$$I(\hbar) = \hbar^{-d/2} e^{S(c)/\hbar} \int_{\mathbf{B}} g(x) e^{-S(x)/\hbar} dx$$

admits an asymptotic power series expansion in $\hbar$:

$$I(\hbar) = A_0 + A_1 \hbar + \cdots + A_m \hbar^m + \cdots \tag{8}$$

Our main question now will be: how to compute the coefficients A_i?

It turns out that although the problem of computing $I(\hbar)$ is transcendental, the problem of computing the coefficients A_i is in fact purely algebraic, and involves only differentiation of the functions S and g at the point c. Indeed, recalling the proof of equation 8 (which we gave in the 1-dimensional case), we see that the calculation of A_i reduces to calculation of integrals of the form

$$\int_V P(x) e^{-B(x,x)/2} dx,$$

where P is a polynomial and B is a positive definite bilinear form (in fact, $B(v,u) = (\partial_v \partial_u S)(c)$). But such integrals can be exactly evaluated. Namely, it is sufficient to consider the case when P is a product of linear functions, in which case the answer is given by the following elementary formula, known to physicists as *Wick's theorem.*

For a positive integer k, consider the set $\{1, \ldots, 2k\}$. By a *pairing* σ on this set we will mean its partition into k disjoint two-element subsets (pairs). A pairing can be visualized by drawing $2k$ points and connecting two points with an edge if they belong to the same pair (see Fig. 1). This will give k edges, which are not connected to each other.

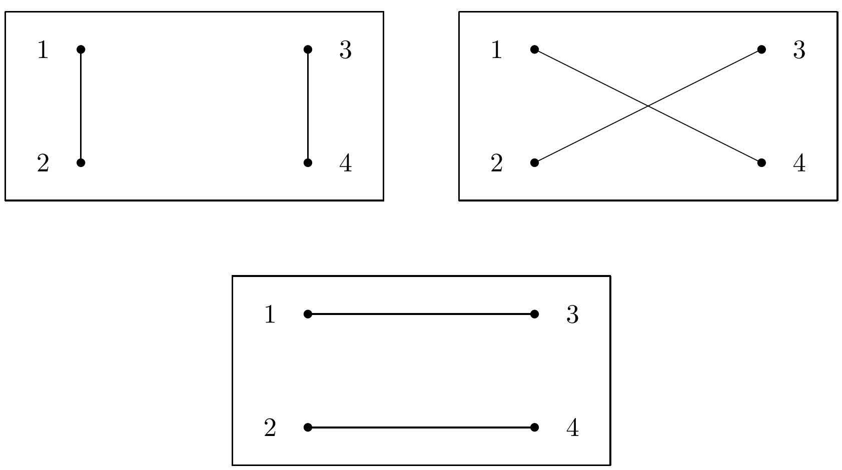


Figure 1. Pairings of the set $\{1, 2, 3, 4\}$

Let us denote the set of pairings on $\{1, \ldots, 2k\}$ by Π_k. It is clear that $|\Pi_k| = \frac{(2n)!}{2^n \cdot n!}$. For any $\sigma \in \Pi_k$, we can think of σ as a permutation of $\{1, \ldots, 2k\}$, such that $\sigma^2 = 1$ and σ has no fixed points. Namely, σ maps any element i to the second element $\sigma(i)$ of the pair containing i.

Theorem 3.1. *Let B^{-1} denote the inverse form on V^*, and let $\ell_1, \ldots, \ell_m \in V^*$. Then, if m is even, we have*

$$\int_V \ell_1(x) \ldots \ell_m(x) e^{-B(x,x)/2} dx = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\sigma \in \Pi_{m/2}} \prod_{i \in \{1,\ldots,m\}/\sigma} B^{-1}(\ell_i, \ell_{\sigma(i)})$$

If m is odd, the integral is zero.

Proof. If m is odd, the statement is obvious, because the integrand is an odd function. So consider the even case. Since both sides of the equation are symmetric polylinear forms in $\ell_1, \dots, \ell_m$, it suffices to prove the result when $\ell_1 = \cdots = \ell_m = \ell$. Further, it is clear that the formula to be proved is stable under linear changes of variable (check it!), so we can choose a coordinate system in such a way that $B(x, x) = x_1^2 + \cdots + x_d^2$, and $\ell(x) = x_1$. Therefore, it is sufficient to assume that $d = 1$, and $\ell(x) = x$. In this case, the theorem says that

$$\int_{-\infty}^{\infty} x^{2k} e^{-x^2/2} dx = (2\pi)^{1/2} \frac{(2k)!}{2^k k!},$$

which is easily obtained from the definition of the Gamma function by change of variable $y = x^2/2$. □

Examples.

$$\int_V \ell_1(x)\ell_2(x)e^{-B(x,x)/2}dx = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} B^{-1}(\ell_1, \ell_2).$$

$$\int_V \ell_1(x)\ell_2(x)\ell_3(x)\ell_4(x)e^{-B(x,x)/2}dx =$$

$$\frac{(2\pi)^{d/2}}{\sqrt{\det B}}(B^{-1}(\ell_1,\ell_2)B^{-1}(\ell_3,\ell_4) + B^{-1}(\ell_1,\ell_3)B^{-1}(\ell_2,\ell_4) + B^{-1}(\ell_1,\ell_4)B^{-1}(\ell_2,\ell_3)).$$

Wick's theorem shows that the problem of computing A_i is of combinatorial nature. In fact, the central role in this computation is played by certain finite graphs, which are called *Feynman diagrams.* They are the main subject of the remainder of this section.

3.2. **Feynman's diagrams and Feynman's theorem.** We come back to the problem of computing the coefficients A_i. Since each particular A_i depends only on a finite number of derivatives of g at c, it suffices to assume that g is a polynomial, or, more specifically, a product of linear functions: $g = \ell_1 \dots \ell_N$, $\ell_i \in V^*$. Thus, it suffices to be able to compute the series expansion of the integral

$$< \ell_1 \dots \ell_N >:= \hbar^{-d/2} e^{S(c)/\hbar} \int_{\mathbf{B}} \ell_1(x) \dots \ell_N(x) e^{-S(x)/\hbar} dx$$

Without loss of generality we may assume that $c = 0$, and $S(c) = 0$. Then the (asymptotic) Taylor expansion of S at c is $S(x) = \frac{B(x,x)}{2} + \sum_{r \geq 3} \frac{B_r(x,x,\dots,x)}{r!}$, where $B_r = d^r f(0)$. Therefore, regarding the left hand side as a power series in $\hbar$, and making a change of variable $x \to x/\sqrt{\hbar}$ (like in the last section), we get

$$< \ell_1 \dots \ell_N >= \hbar^{N/2} \int_V \ell_1(x) \dots \ell_N(x) e^{-\frac{B(x,x)}{2} - \sum_{r\geq 3} \hbar^{r/2-1} \frac{B_r(x,\dots,x)}{r!}} dx.$$

(This is an identity of expansions in $\hbar$, as we ignored the rapidly decaying error which comes from replacing the box by the whole space).

The theorem below, due to Feynman, gives the value of this integral in terms of Feynman diagrams. This theorem is easy to prove but is central in quantum field theory, and will be one of the main theorems of this course. Before formulating this theorem, let us introduce some notation.

Let $G_{\geq 3}(N)$ be the set of isomorphism classes of graphs with N 1-valent "external" vertices, labeled by $1, \dots, N$, and a finite number of unlabeled "internal" vertices, of any valency ≥ 3. Note that here and below graphs are allowed to have multiple edges between two vertices, and loops from a vertex to itself (see Fig. 2).

For each graph $\Gamma \in G_{\geq 3}(N)$, we define the *Feynman amplitude* of Γ as follows.

1. Put the covector ℓ_j at the j-th external vertex.
2. Put the tensor $-B_m$ at each m-valent internal vertex.
3. Take the contraction of the tensors along edges of Γ, using the bilinear form B^{-1}. This will produce a number, called the amplitude of Γ and denoted $F_\Gamma(\ell_1, \dots, \ell_N)$.

Remark. If Γ is not connected, then F_Γ is defined to be the product of numbers obtained from the connected components. Also, the amplitude of the empty diagram is defined to be 1.

Theorem 3.2. *(Feynman) One has*

$$< \ell_1 \dots \ell_N >= \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\Gamma \in G_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

where $b(\Gamma)$ is is the number of edges minus the number of internal vertices of Γ.

(here by an automorphism of Γ we mean a permutation of vertices AND edges which preserves the graph structure, see Fig. 3; thus there can exist nontrivial automorphisms which act trivially on vertices).

Remark 1. Note that this sum is infinite, but $\hbar$-adically convergent.

Remark 2. We note that Theorem 3.2 is a generalization of Wick's theorem: the latter is obtained if $S(x) = B(x, x)/2$. Indeed, in this case graphs which give nonzero amplitudes do not have internal vertices, and thus reduce to graphs corresponding to pairings σ.

Let us now make some comments about the terminology. In quantum field theory, the function $< \ell_1 \dots \ell_N >$ is called the *N-point correlation function*, and graphs Γ are called *Feynman diagrams*. The form B^{-1} which is put on the edges is called *the propagator*. The cubic and higher terms $B_m/m!$ in the expansion of the function S are called *interaction terms*, since such terms (in the action functional) describe interaction between particles. The situation in which S is quadratic (i.e., there is no interaction) is called a *free theory*; i.e. for the free theory the correlation functions are determined by Wick's formula.

Remark 3. Sometimes it is convenient to consider normalized correlation functions $< \ell_1 \dots \ell_N >_{\mathrm{norm}} = < \ell_1 \dots \ell_N > / < \emptyset >$ (where $< \emptyset >$ denotes the integral without insertions). Feynman's theorem

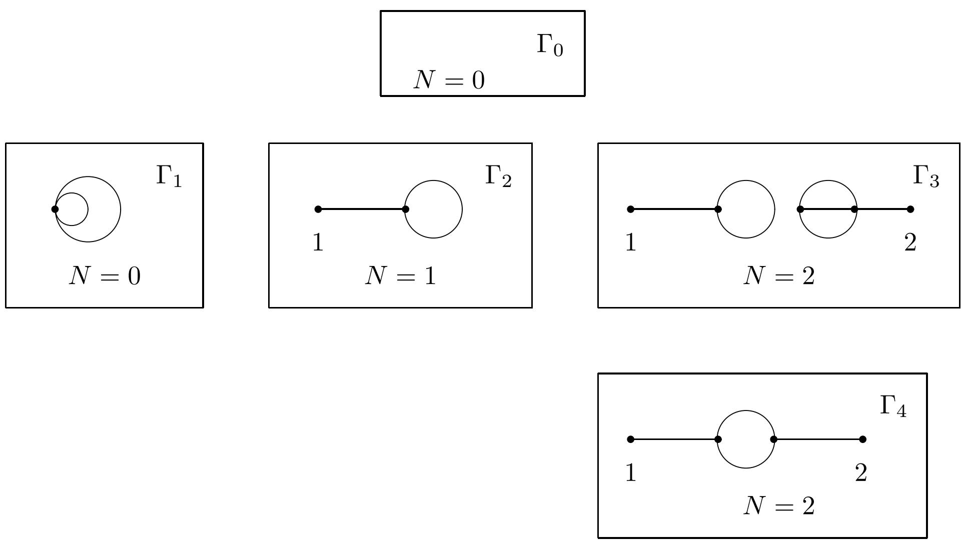


Figure 2. Elements of $G_{\geq 3}(N)$

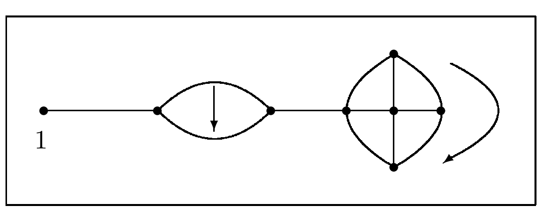


Figure 3. An automorphism of a graph

implies that they are given by the formula

$$< \ell_1 \dots \ell_N >_{\mathrm{norm}} = \sum_{\Gamma \in G^*_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

where $G^*_{\geq 3}(N)$ is the set of all graphs in $G_{\geq 3}(N)$ which have no components without external vertices.

3.3. **Another version of Feynman's theorem.** Before proving Theorem 3.2, we would like to slightly modify and generalize it. Namely, in quantum field theory it is often useful to consider an interacting theory as a deformation of a free theory. This means that $S(x) = B(x,x)/2 + \tilde{S}(x)$, where $\tilde{S}(x)$ is the perturbation $\tilde{S}(x) = \sum_{m \geq 0} g_m B_m(x, x, \dots, x)/m!$, where g_m are (formal) parameters. Consider the *partition function*

$$Z = \hbar^{-d/2} \int_V e^{-S(x)/\hbar} dx$$

as a series in g_i and $\hbar$ (this series involves only positive powers of g_i but arbitrary powers of $\hbar$; however, the coefficient of a given monomial $\prod_i g_i^{n_i}$ is a finite sum, and hence contains only finitely many powers of $\hbar$).

Let $\mathbf{n} = (n_0, n_1, \dots)$ be a sequence of nonnegative integers, almost all zero. Let $G(\mathbf{n})$ denote the set of isomorphism classes of graphs with n_0 0-valent vertices, n_1 1-valent vertices, n_2 2-valent vertices, etc. (thus, now we are considering graphs without external vertices).

Theorem 3.3. *One has*

$$Z = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\mathbf{n}} \left(\prod_i g_i^{n_i} \right) \sum_{\Gamma \in G(\mathbf{n})} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma,$$

where F_Γ is the amplitude defined as before, and $b(\Gamma)$ is the number of edges minus the number of vertices of Γ.

We will prove Theorem 3.3 in the next subsection. Meanwhile, let us show that Theorem 3.2 is in fact a special case of Theorem 3.3. Indeed, because of symmetry of the correlation functions with respect to $\ell_1, \dots, \ell_N$, it is sufficient to consider the case $\ell_1 = \dots = \ell_N = \ell$. In this case, denote the correlation function $< \ell^N >$ (expectation value of ℓ^N). Clearly, to compute $< \ell^N >$ for all N, it is sufficient to compute the generating function $< e^{t\ell} > := \sum < \ell^N > \frac{t^N}{N!}$. But this expectation value is exactly the one given by Theorem 3.3 for $g_i = 1$, $i \geq 3$, $g_0 = g_2 = 0$, $g_1 = -\hbar t$, $B_1 = \ell$, $B_0 = 0$, $B_2 = 0$. Thus, Theorem 3.3 implies Theorem 3.2 (note that the factor $N!$ in the denominator is accounted for by the fact that in Theorem 3.3 we consider unlabeled, rather than labeled, 1-valent vertices – convince yourself of this!).

3.4. **Proof of Feynman's theorem.** Now we will prove Theorem 3.3. Let us make a change of variable $y = x/\sqrt{\hbar}$. Expanding the exponential in a Taylor series, we obtain

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}},$$

where

$$Z_{\mathbf{n}} = \int_V e^{-B(y,y)/2} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} (-\hbar^{i/2-1} B_i(y, y, \dots, y))^{n_i} dy$$

Writing B_i as a sum of products of linear functions, and using Wick's theorem, we find that the value of the integral for each $\mathbf{n}$ can be expressed combinatorially as follows.

1. Attach to each factor $-B_i$ a "flower" — a vertex with i outgoing edges (see Fig. 4).

2. Consider the set T of ends of these outgoing edges (see Fig. 5), and for any pairing σ of this set, consider the corresponding contraction of tensors $-B_i$ using the form B^{-1}. This will produce a number $F(\sigma)$.

3. The integral $Z_{\mathbf{n}}$ is given by

$$Z_{\mathbf{n}} = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \hbar^{n_i(\frac{i}{2}-1)} \sum_\sigma F_\sigma \tag{9}$$

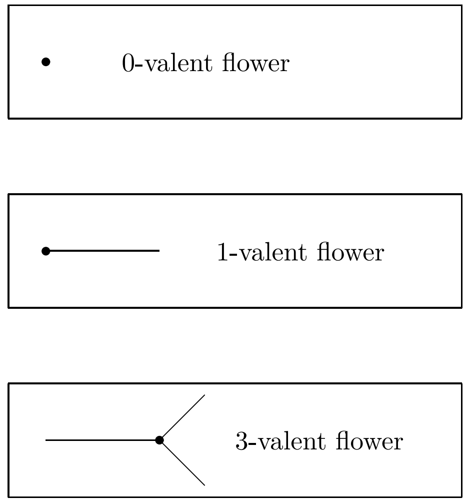


FIGURE 4

Now, recall that pairings on a set can be visualized by drawing its elements as points and connecting them with edges. If we do this with the set T, all ends of outgoing edges will become connected with each other in some way, i.e. we will obtain a certain (unoriented) graph $\Gamma = \Gamma_\sigma$ (see Fig. 6). Moreover, it is easy to see that the number $F(\sigma)$ is nothing but the amplitude F_Γ.

It is clear that any graph Γ with n_i i-valent vertices for each i can be obtained in this way. However, the same graph can be obtained many times, so if we want to collect the terms in the sum over σ, and turn it into a sum over Γ, we must find the number of σ which yield a given Γ.

For this purpose, we will consider the group G of permutations of T, which preserves "flowers" (i.e. endpoints of any two edges outgoing from the same flower end up again in the same flower). This group involves

1) permutations of "flowers" with a given valency;

2) permutation of the i edges inside each i-valent "flower".

More precisely, the group G is the semidirect product $(\prod_i S_{n_i}) \ltimes (\prod_i S_i^{n_i})$. Note that $|G| = \prod_i (i!)^{n_i} n_i!$, which is the product of the numbers in the denominator of the formula (9).

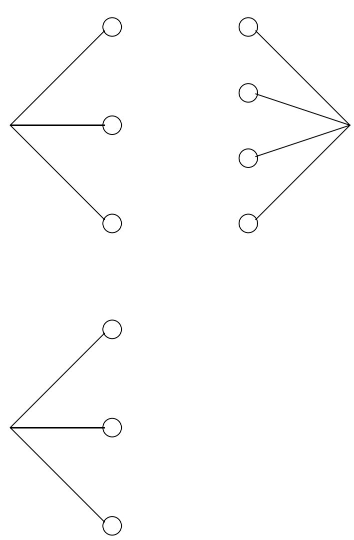


FIGURE 5. The set T for $\vec{n} = (0, 0, 0, 2, 1, 0, 0, \ldots)$ (the set of white circles)

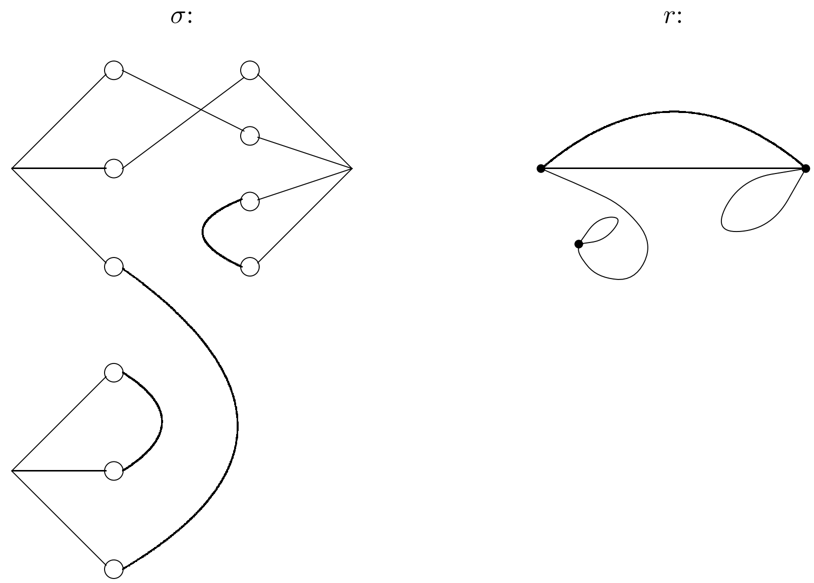


FIGURE 6. A pairing σ of T and the corresponding graph Γ.

The group G acts on the set of all pairings σ of T. Moreover, it acts transitively on the set P_Γ of pairings of T which yield a given graph Γ. Moreover, it is easy to see that the stabilizer of a given pairing is $\mathrm{Aut}(\Gamma)$. Thus, the number of pairings giving Γ is

$$\frac{\prod_i (i!)^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|}.$$

Hence,

$$\sum_\sigma F_\sigma = \sum_\Gamma \frac{\prod_i (i!)^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|} F_\Gamma.$$

Finally, note that the exponent of $\hbar$ in equation (9) is $\sum_i (i/2-1)$, which is the number of edges of Γ minus the number of vertices, i.e. $b(\Gamma)$. Substituting this into (9), we get the result.

Example. Let $d=1$, $V=\mathbb{R}$, $g_i=g$, $B_i=-z^i$ for all i (where z is a formal variable), $\hbar=1$. Then we find the asymptotic expansion

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+ge^{zx}} = \sum_{n\geq 0} g^n \sum_{\Gamma\in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|},$$

where $G(n,k)$ is the set of isomorphism classes of graphs with n vertices and k edges. Expanding the left hand side, we get

$$\sum_k \sum_{\Gamma\in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|} = \frac{e^{z^2n^2/2}}{n!},$$

and hence

$$\sum_{\Gamma\in G(n,k)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k! n!},$$

Exercise. Check this by direct combinatorics.

3.5. **Sum over connected diagrams.** Now we will show that the logarithm of the partition function Z is also given by summation over diagrams, but with only *connected diagrams* taken into account. This significantly simplifies the analysis of Z in the first few orders of perturbation theory, since the number of connected diagrams with a given number of vertices and edges is significantly smaller than the number of all diagrams.

Theorem 3.4. *Let $Z_0 = \frac{(2\pi)^{d/2}}{\det(B)}$. Then one has*

$$\ln(Z/Z_0) = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G_c(\mathbf{n})} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma,$$

where $G_c(\mathbf{n})$ is the set of connected graphs in $G(\mathbf{n})$.

Remark. We agree that the empty graph is not connected.

Proof. For any graphs Γ_1, Γ_2, let $\Gamma_1\Gamma_2$ stand for the disjoint union of Γ_1 and Γ_2, and for any graph Γ let Γ^n denote the disjoint union of n copies of Γ. Then every graph can be uniquely written as $\Gamma_1^{k_1} \dots \Gamma_l^{k_l}$, where Γ_j are connected non-isomorphic graphs. Moreover, it is clear that $F_{\Gamma_1\Gamma_2} = F_{\Gamma_1} F_{\Gamma_2}$, $b(\Gamma_1\Gamma_2) = b(\Gamma_1) + b(\Gamma_2)$, and $|\mathrm{Aut}(\Gamma_1^{k_1} \dots \Gamma_l^{k_l})| = \prod_j (|(\mathrm{Aut}(\Gamma_j)|^{k_j} k_j!)$. Thus, exponentiating the equation of Theorem 3.4, and using the above facts together with the Taylor series for the function e^x, we arrive at Theorem 3.3. As the Theorem 3.3 has been proved, so is Theorem 3.4 □

3.6. **Loop expansion.** It is very important to note that since summation in Theorem 3.4 is over connected Feynman diagrams, the number $b(\Gamma)$ is the number of loops in Γ minus 1. In particular, the lowest coefficient in $\hbar$ is that of $\hbar^{-1}$, and it is the sum over all trees; the next coefficient is of $\hbar^0$, and it is the sum over all diagrams with one loop (cycle); the next coefficient to $\hbar$ is the sum over two-loop diagrams, and so on. Therefore, physicists refer to the expansion of Theorem 3.4 as *loop expansion.*

Let us study the two most singular terms in this expansion (with respect to $\hbar$), i.e. the terms given by the sum over trees and 1-loop graphs.

Let x_0 be the critical point of the function S. It exists and is unique, since g_i are assumed to be formal parameters. Let $G^{(j)}(\mathbf{n})$ denote the set of classes of graphs in $G_c(\mathbf{n})$ with j loops. Let

$$(\ln(Z/Z_0))_j = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G^{(j)}(\mathbf{n})} \frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|}.$$

Theorem 3.5.

$$(\ln(Z/Z_0))_0 = -S(x_0), \tag{10}$$

and

$$(\ln(Z/Z_0))_1 = \frac{1}{2} \ln \frac{\det(B)}{\det S''(x_0)}. \tag{11}$$

Proof. First note that the statement is purely combinatorial. This means, in particular, that it is sufficient to check that the statement yields the correct asymptotic expansion of the right hand sides of equations (10),(11). in the case when S is a polynomial with real coefficients of the form $B(x,x)/2 + \sum_{i=0}^N g_i B_i(x, x, \dots, x)/i!$. To do this, let $Z = \hbar^{-d/2} \int_{\mathbf{B}} e^{-S(x)/\hbar}$, where $\mathbf{B}$ is a sufficiently small box around 0. For sufficiently small g_i, the function S has a unique global maximum point x_0 in $\mathbf{B}$, which is nondegenerate. Thus, by the steepest descent formula, we have

$$Z/Z_0 = e^{-S(x_0)/\hbar} I(\hbar),$$

where $I(\hbar) = \sqrt{\frac{\det(B)}{\det S''(x_0)}}(1 + a_1\hbar + a_2\hbar^2 + \cdots)$ (asymptotically). Thus,

$$\ln(Z/Z_0) = -S(x_0)\hbar^{-1} + \frac{1}{2} \ln \frac{\det(B)}{\det S''(x_0)} + O(\hbar).$$

This implies the result. □

Physicists call the expression $(\ln(Z/Z_0))_0$ the *classical (or tree) approximation* to the quantum mechanical quantity $\hbar \ln(Z/Z_0)$, and the sum $(\ln(Z/Z_0))_0 + \hbar(\ln(Z/Z_0))_1$ the *one loop approximation.* Similarly one defines higher loop approximations. Note that the classical approximation is obtained by finding the critical point and value of the classical action $S(x)$, which in the mechanics and field theory situation corresponds to solving the classical equations of motion.

3.7. **Nonlinear equations and trees.** As we have noted, Theorem 3.5 does not involve integrals and is purely combinatorial. Therefore, there should exist a purely combinatorial proof of this theorem. Such a proof indeed exists. Here we will give a combinatorial proof of the first statement of the Theorem (formula (10)).

Consider the equation $S'(x) = 0$, defining the critical point x_0. This equation can be written as $x = \beta(x)$, where

$$\beta(x) := -\sum_{i\geq 1} g_i \hat{B}^{-1} B_i(x, x, \ldots, x, ?)/(i-1)!,$$

where $\hat{B}^{-1} : V^* \to V$ is the operator corresponding to the form B^{-1}.

In the sense of power series norm, β is a contracting mapping. Thus, $x_0 = \lim_{N\to\infty} \beta^N(x)$, for any initial vector $x \in V$. In other words, we will obtain x_0 if we keep substituting the series $\beta(x)$ into itself. This leads to summation over trees (explain why!). More precisely, we get the following expression for x_0:

$$x_0 = \sum_{\mathbf{n}} (\prod g_i^{n_i}) \sum_{\Gamma\in G^{(0)}(\mathbf{n},1)} \frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

where $G^{(0)}(\mathbf{n}, 1)$ is the set of trees with one external vertex and n_i internal vertices of degree i. Now, since $S(x) = B(x, x)/2 + \sum g_i B_i(x, x, \ldots, x)/i!$, the expression $-S(x_0)$ equals the sum of expressions $(\prod g_i^{n_i}) \frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|}$ over all trees (without external vertices). Indeed, the term $B(x_0, x_0)/2$ corresponds to gluing two trees with external vertices (identifying the two external vertices, so that they disappear); so it corresponds to summing over trees with a marked edge, i.e. counting each tree as many times as it has edges. On the other hand, the term $g_i B_i(x_0, \ldots, x_0)/i!$ corresponds to gluing i trees with external vertices together at the external vertices (making a tree with a marked vertex). So $\sum g_i B_i(x_0, \ldots, x_0)/i!$ corresponds to summing over trees with a marked vertex, i.e. counting each trees as many times as it has vertices. But the number of vertices of a tree exceeds the number of edges by 1. Thus, the difference $-S(x_0)$ of the above two contributions corresponds to summing over trees, counting each exactly once. This implies formula (10).

3.8. **Counting trees and Cayley's theorem.** In this section we will apply Theorem 3.5 to tree counting problems, in particular will prove a classical theorem due to Cayley that the number of labeled trees with n vertices is n^{n-2}.

We consider essentially the same example as we considered above: $d = 1$, $B_i = -1$, $g_i = g$. Thus, we have $S(x) = \frac{x^2}{2} - ge^x$. By Theorem 3.5, we have

$$\sum_{n\geq 0} g^n \sum_{\Gamma\in T(n)} \frac{1}{|Aut(\Gamma)|} = -S(x_0),$$

where $T(n)$ is the set of isomorphism classes of trees with n vertices, and x_0 is the root of the equation $S'(x) = 0$, i.e. $x = ge^x$.

In other words, let $f(z)$ be the function inverse to xe^{-x} near $x = 0$. Then we have $x_0 = f(g)$. Thus, let us find the Taylor expansion of f. This is given by the following classical result.

Proposition 3.6. *One has*

$$f(g) = \sum_{n\geq 1} \frac{n^{n-2}}{(n-1)!} g^n.$$

Proof. Let $f(g) = \sum_{n\geq 1} a_n g^n$. Then

$$a_n = \frac{1}{2\pi i} \oint \frac{f(g)}{g^{n+1}} dg = \frac{1}{2\pi i} \oint \frac{x}{(xe^{-x})^{n+1}} d(xe^{-x}) =$$

$$\frac{1}{2\pi i} \oint e^{nx} \frac{1-x}{x^n} dx = \frac{n^{n-1}}{(n-1)!} - \frac{n^{n-2}}{(n-2)!} = \frac{n^{n-2}}{(n-1)!}.$$

□

Now we find
$$-S(x_0) = -f(g)^2/2 + ge^{f(g)}.$$
Thus
$$-(d/dg)S(x_0) = -f(g)f'(g) + ge^{f(g)}f'(g) + e^{f(g)} = e^{f(g)} = \frac{f(g)}{g}.$$
This means that
$$-S(x_0) = \int_0^g \frac{f(a)}{a}da = \sum_{n\geq 1} \frac{n^{n-2}}{n!}g^n.$$
This shows that
$$\sum_{\Gamma\in T(n)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{n-2}}{n!}$$
But each isomorphism class of unlabeled trees with n vertices has $\frac{n!}{|\mathrm{Aut}(\Gamma)|}$ nonisomorphic labelings. Thus the latter formula implies

Corollary 3.7. *(A. Cayley) The number of labeled trees with n vertices is n^{n-2}.*

3.9. **Counting trees with conditions.** In a similar way we can count labeled trees with conditions on vertices. For example, let us compute the number of labeled trivalent trees with m vertices (i.e. trees that have only 1-valent and 3-valent vertices). Clearly, $m = 2k$, otherwise there is no such trees. The relevant action functional is $S(x) = \frac{x^2}{2} - g(x + x^3/6)$. Then the critical point x_0 is obtained from the equation $g(x^2/2 + 1) - x = 0$, which yields $x_0 = \frac{1-\sqrt{1-2g^2}}{g}$. Thus, the tree sum $(\ln(Z/Z_0))_0$ equals
$$(\ln(Z/Z_0))_0 = -S(x_0) = \frac{1-(1-2g^2)^{3/2}}{3g^2} - 1.$$
Expanding this in a Taylor series, we find
$$(\ln(Z/Z_0))_0 = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot \cdots \cdot (2n-1)}{(n+2)!}g^{2n+2}$$
Hence, we get

Corollary 3.8. *The number of trivalent labeled trees with $m = 2k$ vertices is $(2k-3)!!\frac{(2k)!}{(k+1)!}$.*

3.10. **Counting oriented trees.** Feynman calculus can be used to count not only non-oriented, but also oriented graphs. For example, suppose we want to count labeled oriented trees, whose vertices are either sources or sinks (see Fig. 7). In this case, it is easy to see (check it!) that the relevant integration problem is in two dimensions, with the action $S = xy - be^x - ae^y$. So the critical point is found from the equations
$$xe^{-y} = a, ye^{-x} = b.$$

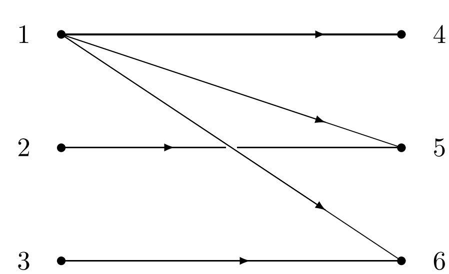


FIGURE 7. A labeled oriented tree with 3 sources and 3 sinks.

Like before, look for a solution $(x, y) = (x_0, y_0)$ in the form

$$x = a + \sum_{p\geq 1, q\geq 1} c_{pq} a^p b^q, y = b + \sum_{p\geq 1, q\geq 1} d_{pq} a^p b^q.$$

A calculation with residues similar to the one we did for unoriented trees yields

$$c_{pq} = \frac{1}{(2\pi i)^2} \oint \oint \frac{x}{a^{p+1} b^{q+1}} da \wedge db = \frac{1}{(2\pi i)^2} \oint \oint \frac{e^{qx+py}}{x^p y^{q+1}} (1 - xy) dx \wedge dy = \frac{q^{p-1} p^{q-1}}{(p-1)! q!}.$$

Similarly, $d_{pq} = \frac{q^{p-1} p^{q-1}}{p!(q-1)!}$. Now, similarly to the unoriented case, we find that $-a\partial_a S(x, y) = x$, $-b\partial_b S(x, y) = y$, so

$$-S(x, y) = b + \int_0^a \frac{x}{u} du = a + b + \sum_{p,q\geq 1} \frac{p^{q-1} q^{p-1}}{p! q!} a^p b^q$$

This implies that the number of labeled trees with p sources and q sinks is$p^{q-1} q^{p-1} \frac{(p+q)!}{p!q!}$. In particular, if we specify which vertices are sources and which are sinks, the number of trees is $p^{q-1} q^{p-1}$.

3.11. **1-particle irreducible diagrams and the effective action.** Let $Z = Z_S$ be the partition function corresponding to the action S. In the previous subsections we have seen that the "classical" (or "tree") part $(\ln(Z_S/Z_0))_0$ of the quantity $\ln(Z_S/Z_0)$ is quite elementary to compute – it is just minus the critical value of the action $S(x)$. Thus, if we could find a new "effective" action S_{eff} (a "deformation" of S) such that

$$(\ln(Z_{\mathrm{S}_{\text{eff}}}/Z_0))_0 = \ln(Z_S/Z_0)$$

(i.e. the classical answer for the effective action is the quantum answer for the original one), then we can regard the quantum theory for the action S as solved. In other words, the problem of solving the quantum theory attached to S (i.e. finding the corresponding integrals) essentially reduces to the problem of computing the effective action S_{eff}.

We will now give a recipe of computing the effective action in terms of amplitudes of Feynman diagrams.

Definition 3.9. An edge e of a connected graph Γ is said to be a bridge, if the graph $\Gamma \setminus\ e$ is disconnected. A connected graph without bridges is called 1-particle irreducible (1PI).

Remark. This is the physical terminology. The mathematical terminology is "2-connected".

To compute the effective action, we will need to consider graphs with external edges (but having at least one internal vertex). Such a graph Γ (with N external edges) will be called 1-particle irreducible if so is the corresponding "amputated" graph (i.e. the graph obtained from Γ by removal of the external edges). In particular, a graph with one internal vertex is always 1-particle irreducible (see Fig. 8), while a single edge graph without internal vertices is defined *not* to be 1-particle irreducible.

Let us denote by $G_{1-\text{irr}}(\mathbf{n}, N)$ the set of isomorphism classes of 1-particle irreducible graphs which N external edges and n_i i-valent internal vertices for each i (where isomorphisms are not allowed to move external edges).

Theorem 3.10. *The effective action S_{eff} is given by the formula*

$$S_{\text{eff}}(x) = \frac{B(x, x)}{2} - \sum_{i\geq 0} \frac{\mathcal{B}_i}{i!},$$

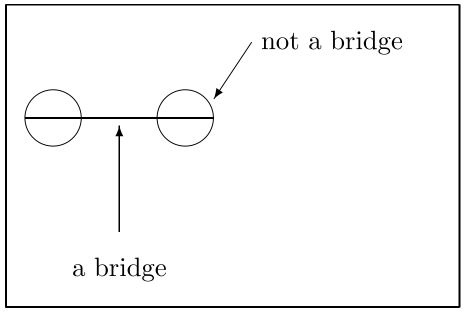
where

$$\mathcal{B}_N(x, x, \ldots, x) = \sum_{\mathbf{n}} (\prod_i g_i^{n_i}) \sum_{\Gamma \in G_{1-\text{irr}}(\mathbf{n}, N)} \frac{\hbar^{b(\Gamma)+1}}{|\text{Aut}(\Gamma)|} F_\Gamma(x_*, x_*, \ldots, x_*),$$

where $x_* \in V^*$ is defined by $x_*(y) := B(x, y)$

Thus, $S_{\text{eff}} = S + \hbar S_1 + \hbar^2 S_2 + ..$ The expressions $\hbar^j S_j$ are called the j-loop corrections to the effective action.

This theorem allows physicists to worry only about 1-particle irreducible diagrams, and is the reason why you will rarely see other diagrams in a QFT textbook. As before, it is very useful in doing low



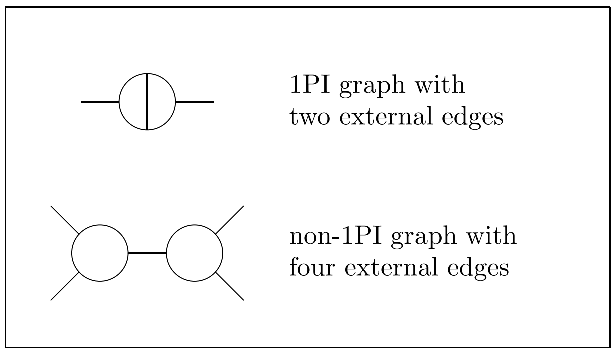


FIGURE 8

order computations, since the number of 1-particle irreducible diagrams with a given number of loops is much smaller than the number of connected diagrams with the same number of loops.

Proof. The proof is based on the following lemma from graph theory.

Lemma 3.11. *Any connected graph* Γ *can be uniquely represented as a tree, whose vertices are 1-particle irreducible subgraphs (with external edges), and edges are the bridges of* Γ.

The lemma is obvious. Namely, let us remove all bridges from Γ. Then Γ will turn into a union of 1-particle irreducible graphs, which should be taken to be the vertices of the said tree.

The tree corresponding to the graph Γ is called the skeleton of Γ (see Fig. 9).

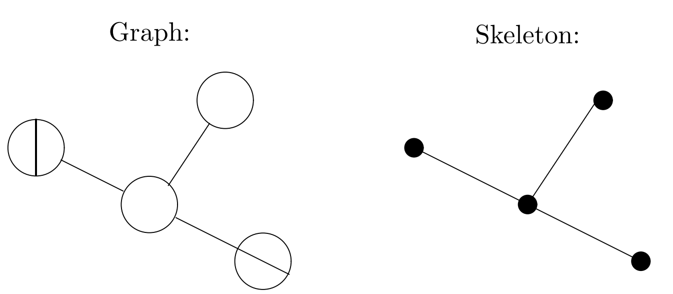


FIGURE 9. The skeleton of a graph.

It is easy to see that the lemma implies the theorem. Indeed, it implies that the sum over all connected graphs occurring in the expression of $\ln(Z_S/Z_0)$ can be written as a sum over skeleton trees, so that the contribution from each tree is (proportional to) the contraction of tensors $\mathcal{B}_i$ put in its vertices, and $\mathcal{B}_i$ is the (weighted) sum of amplitudes of all 1-particle irreducible graphs with i external edges. $\square$

3.12. **1-particle irreducible graphs and the Legendre transform.** Recall the notion of Legendre transform.

Let f be a smooth function on a vector space Y, such that the map $Y \to Y^*$ given by $x \to df(x)$ is a diffeomorphism. Then one can define the Legendre transform of f as follows. For $p \in Y^*$, let $x_0(p)$ be the critical point of the function $(p, x) - f(x)$ (i.e. the unique solution of the equation $df(x) = p$). Then the Legendre transform of f is the function on Y^* defined by

$$L(f)(p) = (p, x_0) - f(x_0).$$

It is easy to see that the differential of $L(f)$ is also a diffeomorphism $Y^* \to Y$ (in fact, inverse to $df(x)$), and that $L^2(f) = f$.

Example. Let $f(x) = ax^2/2$. Then $px - f = px - x^2/2$ has a critical point at $p = x/a$, and the critical value is $p^2/2a$. $L(ax^2/2) = p^2/2a$. Similarly, if $f(x) = B(x, x)/2$ where B is a nondegenerate symmetric form, then $L(f)(p) = B^{-1}(p, p)/2$.

Now let us consider Theorem 3.10 in the situation of Theorem 3.2. Thus, $S(x) = B(x, x)/2 + O(x^3)$, and we look at

$$Z(J) = \hbar^{-d/2} \int_V e^{\frac{J \cdot x - S(x)}{\hbar}} dx$$

By Theorem 3.10, one has

$$\ln(Z(J)/Z_0) = -S_{\text{eff}}(x_0, J),$$

where the effective action $S_{\text{eff}}(x, J)$ given by summation over 1-particle irreducible graphs.

Now, we must have $S_{\text{eff}}(x, J) = -J \cdot x + S_{\text{eff}}(x)$, since the only 1PI graph which contains 1-valent internal vertices (corresponding to J) is the graph with one edge, connecting an internal vertex with an external one (so it yields the term $-J \cdot x$, and other graphs contain no J-vertices). This shows that $\ln(Z(J)/Z_0)$ is the critical value of $J \cdot x - S_{\text{eff}}(x)$. Thus we have proved the following.

Proposition 3.12. *We have*

$$S_{\text{eff}}(x) = L(\ln(Z(J)/Z_0)),\ \ln(Z(J)/Z_0) = L(S_{\text{eff}}(x)).$$

Physicists formulate this result as follows: the effective action is the Legendre transform of the logarithm of the generating function for quantum correlators (and vice versa).