A. Introduction

ps1 A1. *Ramsey's theorem*

(a) Let s and r be positive integers. Show that there is some integer $n = n(s, r)$ so that if every edge of the complete graph K_n on n vertices is colored with one of r colors, then there is a monochromatic copy of K_s.

(b) Let $s \geq 3$ be a positive integer. Show that if the edges of the complete graph on $\binom{2s-2}{s-1}$ vertices are colored with 2 colors, then there is a monochromatic copy of K_s.

ps1 A2. Prove that it is possible to color $\mathbb{N}$ using two colors so that there is no infinitely long monochromatic arithmetic progression.

ps1 A3. *Many monochromatic triangles*

(a) True or false: If the edges of K_n are colored using 2 colors, then at least $1/4 - o(1)$ fraction of all triangles are monochromatic. (Note that $1/4$ is the fraction one expects if the edges were colored uniformly at random.)

(b) True or false: if the edges of K_n are colored using 3 colors, then at least $1/9 - o(1)$ fraction of all triangles are monochromatic.

(c) ($\star$ do not submit) True or false: if the edges of K_n are colored using 2 colors, then at least $1/32 - o(1)$ fraction of all copies of K_4's are monochromatic.

(d) (do not submit) Prove that for every s and r, there is some constant $c > 0$ so that for every sufficiently large n, if the edges of K_n are colored using r colors, then at least c fraction of all copies of K_s are monochromatic.

B. Forbidding subgraphs

ps1 B1. Show that a graph with n vertices and m edges has at least $\frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$ triangles.

ps1$\star$ B2. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles.

ps1$\star$ B3. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains some edge in at least $(1/6 - o(1))n$ triangles, and that this constant $1/6$ is best possible.

B4. *K_{r+1}-free graphs close to the Turán bound are nearly r-partite*

ps1 (a) Let G be an n-vertex triangle-free graph with at least $\lfloor n^2/4 \rfloor - k$ edges. Prove that G can be made bipartite by removing at most k edges.

ps1$\star$ (b) Let G be an n-vertex K_{r+1}-free graph with at least $e(T_{n,r}) - k$ edges, where $T_{n,r}$ is the Turán graph. Prove that G can be made r-partite by removing at most k edges.

B5. Let G be a K_{r+1}-free graph. Prove that there is another graph H on the same vertex set as G such that $\chi(H) \leq r$ and $d_H(x) \geq d_G(x)$ for every vertex x (here $d_H(x)$ is the degree of x in H, and likewise with $d_G(x)$ for G). Give another proof of Turán's theorem from this fact.

B6. *Turán density.* Let H be a r-uniform hypergraph, let its *Turán number* $\mathrm{ex}^{(r)}(n, H)$ be the maximum number of edges in an r-uniform hypergraph on n vertices that does not contain H as a subgraph. Prove that the fraction $\mathrm{ex}^{(r)}(n, H)/\binom{n}{r}$ is a nonincreasing function of n, so that it has a limit $\pi(H)$ as $n \to \infty$, called the *Turán density* of H.

ps1 B7. *Supersaturation.* Let H be a graph and ρ a constant such that $\limsup_{n\to\infty} \operatorname{ex}(n,H)/\binom{n}{2} \le \rho$. Prove that for every $\epsilon > 0$ there exists some constant $c = c(H,\epsilon) > 0$ such that for sufficiently large n, every n-vertex graph with at least $(\rho+\epsilon)\binom{n}{2}$ edges contains at least $cn^{v(H)}$ copies of H.

ps1 B8. Let S be a set of n points in the plane, with the property that no two points are at distance greater than 1. Show that S has at most $\lfloor n^2/3 \rfloor$ pairs of points at distance greater than $1/\sqrt{2}$. Also, show that the bound $\lfloor n^2/3 \rfloor$ is tight (i.e., cannot be improved).

ps1 B9. (How *not* to define density in a product set) Let $S \subset \mathbb{Z}^2$. Define

$$d_k(S) = \max_{\substack{A,B\subset\mathbb{Z}\\ |A|=|B|=k}} \frac{|S\cap(A\times B)|}{|A||B|}.$$

Show that $\lim_{k\to\infty} d_k(S)$ exists and is always either 0 or 1.

B10. Show that, for every $\epsilon > 0$, there exists $\delta > 0$ such that every graph with n vertices and at least ϵn^2 edges contains a copy of $K_{s,t}$ where $s \ge \delta \log n$ and $t \ge n^{0.99}$.

ps2 B11. *Density version of Kővári–Sós–Turán.* Prove that for every positive integers $s \le t$, there are constants $C, c > 0$ such that every n-vertex graph with $p\binom{n}{2}$ edges contains at least $cp^{st}n^{s+t}$ copies of $K_{s,t}$, provided that $p \ge Cn^{-1/s}$.

ps2⋆ B12. *Hypergraph Kővári–Sós–Turán and a proof of Erdős–Stone–Simonovits*

(a) Prove that for every positive integer t there is some C so that every 3-uniform hypergraph on n vertices and at least $Cn^{3-t^{-2}}$ edges (i.e., triples) contains a copy of $K^{(3)}_{t,t,t}$, the complete tripartite 3-uniform hypergraph with t vertices in each part.

(b) Deduce that $\operatorname{ex}(n,H) \le (\frac{1}{4}+o(1))n^2$ for every graph H with $\chi(H) \le 3$.

(c) Explain how to generalize the above strategy to prove the Erdős–Stone–Simonovits theorem for every H (sketch the key steps).

ps2 B13. Find a graph H with $\chi(H) = 3$ and $\operatorname{ex}(n,H) > \frac{1}{4}n^2 + n^{1.99}$ for all sufficiently large n.

ps2⋆ B14. *Construction of a C_6-free graph.* Let q be an odd prime power. Let S denote the quadratic surface in the 4-dimensional projective space over $\mathbb{F}_q$ (whose points are nonzero points of $\mathbb{F}_q^5$ modulo the equivalence relation $(x_0,x_1,x_2,x_3,x_4) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4)$ for $\lambda \in \mathbb{F}_q^\times$) given by the equation (you may use another quadratic form if you wish)

$$x_0^2 + 2x_1x_2 + 2x_3x_4 = 0.$$

Let $\mathcal{L}$ be the set of lines contained in S.

(a) Prove that no three lines of $\mathcal{L}$ lie in the same plane.

(b) Show that the point-line incidence bipartite graph between S and $\mathcal{L}$ is a $(q+1)$-regular graph on $2(q^3+q^2+q+1)$ vertices with no cycles of length at most 6. Conclude that $\operatorname{ex}(n,C_6) \ge cn^{4/3}$ for some constant $c > 0$.

The next two problems concern the dependent random choice technique.

ps2 B15. Let $\epsilon > 0$. Show that, for sufficiently large n, every K_4-free graph with n vertices and at least ϵn^2 edges contains an independent set of size at least $n^{1-\epsilon}$.

ps2⋆ B16. *Extremal numbers of degenerate graphs*

(a) Prove that there is some absolute constant $c > 0$ so that for every positive integer r, every n-vertex graph with at least $n^{2-c/r}$ edges contains disjoint vertex subsets A and B such that every subset of r vertices in A has at least n^c neighbors in B and every subset of r vertices in B has at least n^c neighbors in A.

(b) We say that a graph H is *r-degenerate* if its vertices can be ordered so that every vertex has at most r neighbors that appear before it in the ordering. Show that for every r-degenerate bipartite graph H there is some constant $C > 0$ so that $\operatorname{ex}(n, H) \le Cn^{2-c/r}$, where c is the same absolute constant from part (a) (c should not depend on H or r).

ps2 B17. Let T be a tree with k edges. Show that $\operatorname{ex}(n, T) \le kn$.

ps2⋆ B18. Show that every n-vertex triangle-free graph with minimum degree greater than $2n/5$ is bipartite.

C. Szemerédi's regularity lemma and applications

For simplicity, you are welcome to apply the equitable version of Szemerédi's regularity lemma.

C1. Let G be a graph and $X, Y \subset V(G)$. If (X, Y) is an $\epsilon\eta$-regular pair, then (X', Y') is ϵ-regular for all $X' \subset X$ with $|X'| \ge \eta|X|$ and $Y' \subset Y$ with $|Y'| \ge \eta|Y|$.

C2. Let G be a graph and $X, Y \subset V(G)$. Say that (X, Y) is *ϵ-homogeneous* if for all $A \subset X$ and $B \subset Y$, one has

$$|e(A, B) - |A|\,|B|\,d(X, Y)| \le \epsilon\,|X|\,|Y|\,.$$

Show that if (X, Y) is ϵ-regular, then it is ϵ-homogeneous. Also, show that if (X, Y) is ϵ^3-homogeneous, then it is ϵ-regular.

ps2 C3. *Unavoidability of irregular pairs.* Let the *half-graph* H_n be the bipartite graph on $2n$ vertices $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ with edges $\{a_i b_j : i \le j\}$.

(a) For every $\epsilon > 0$, explicitly construct an ϵ-regular partition of H_n into $O(1/\epsilon)$ parts.

(b) Show that there is some $c > 0$ such that for every $\epsilon \in (0, c)$, every integer k and sufficiently large multiple n of k, every partition of the vertices of H_n into k equal-sized parts contains at least ck pairs of parts which are not ϵ-regular.

ps2 C4. Show that there is some absolute constant $C > 0$ such that for every $0 < \epsilon < 1/2$, every graph on n vertices contains an ϵ-regular pair of vertex subsets each with size at least δn, where $\delta = 2^{-\epsilon^{-C}}$.

C5. *Existence of a regular set.* Given a graph G, we say that $X \subset V(G)$ is *ϵ-regular* if the pair (X, X) is ϵ-regular, i.e., for all $A, B \subset X$ with $|A|, |B| \ge \epsilon|X|$, one has $|d(A, B) - d(X, X)| \le \epsilon$. This problem asks for two different proofs of the claim: for every $\epsilon > 0$, there exists $\delta > 0$ such that every graph contains an ϵ-regular subset of vertices of size at least δ fraction of the vertex set.

ps3 (a) Prove the claim using Szemerédi's regularity lemma, showing that one can obtain the ϵ-regular subset by combining a suitable sub-collection of parts from a regular partition.

ps3⋆ (b) Give an alternative proof of the claim showing that one can take $\delta = \exp(-\exp(\epsilon^{-C}))$ for some constant C.

ps3 C6. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex K_4-free graph with at least $(\frac{1}{8} + \epsilon)n^2$ edges contains an independent set of size at least δn.

C7. Show that for ever $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex K_4-free graph with at least $(\frac{1}{8} - \delta)n^2$ edges and independence number at most δn can be made bipartite by removing at most ϵn^2 edges.

ps3 C8. Show that the number of non-isomorphic n-vertex triangle-free graphs is $2^{(1/4+o(1))n^2}$.

C9. Show that for every H there exists some $\delta > 0$ such that for all sufficiently large n, if G is an n-vertex graph with average degree at least $(1 - \delta)n$ and the edges of G are colored using 2 colors, then there is a monochromatic copy of H.

ps3 C10. Show that for every H and $\epsilon > 0$ there exists $\delta > 0$ such that every graph on n vertices without an induced copy of H contains an induced subgraph on at least δn vertices whose edge density is at most ϵ or at least $1 - \epsilon$.

C11. *Random graphs are ϵ-regular.* Let G be a random bipartite graph between disjoint sets of vertices X and Y with $|X| = |Y| = n$, such that every pair in $X \times Y$ appears as an edge of G independently with the same probability. Show that there is some absolute constant $c > 0$ such that with probability at least $1 - e^{-n^{1+c}}$ for sufficiently large n, the pair (X, Y) is ϵ-regular in G with $\epsilon = n^{-c}$.

(You may use the following special case of the Azuma–Hoeffding inequality: if $X_1, \dots, X_N$ are independent random variables taking values in $[-1, 1]$, and $S = X_1 + \cdots + X_N$, then $\mathbb{P}(S \geq \mathbb{E}S + t) \leq e^{-t^2/(2N)}$.)

ps3⋆ C12. Show that for every graph H there is some graph G such that if the edges of G are colored with two colors, then some induced subgraph of G is a monochromatic copy of H.

ps3⋆ C13. Show that for every $c > 0$, there exists $c' > 0$ such that every graph on n vertices with at least cn^2 edges contains a d-regular subgraph with $d \geq c'n$ (here *d-regular* refers to every vertex having degree d).

ps4 C14. Show that there is a constant $c > 0$ so that for every sufficiently small $\epsilon > 0$ and sufficiently large $n > n_0(\epsilon)$ there exists an n-vertex graph with at most $\epsilon^{c \log(1/\epsilon)} n^3$ triangles that cannot be made triangle-free by removing fewer than ϵn^2 edges. (In particular, this shows that one cannot take $\delta = \epsilon^C$ for some constant $C > 0$ in the triangle removal lemma.)

C15. *Removal lemma for bipartite graphs with polynomial bounds.* Prove that for every bipartite graph H, there is a constant C such that for every $\epsilon > 0$, every n-vertex graph with fewer than $\epsilon^C n^{v(H)}$ copies of H can be made H-free by removing at most ϵn^2 edges.

ps4 C16. Let H be a n-vertex 3-uniform hypergraph such that every 6 vertices contain strictly fewer than 3 triples. Prove that H has $o(n^2)$ edges.

(Hint in white:)

ps4 C17. Assuming the tetrahedron removal lemma for 3-uniform hypergraphs, deduce that if $A \subset [N]^2$ contains no axes-aligned squares (i.e., four points of the form $(x, y), (x + d, y), (x, y + d), (x + d, y + d)$, where $d \neq 0$), then $|A| = o(N^2)$.

ps4⋆ C18. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset [n]$ has fewer than δn^2 many triples $(x, y, z) \in A^3$ with $x + y = z$, then there is some $B \subset A$ with $|A \setminus B| \leq \epsilon n$ such that B is sum-free, i.e., there do not exist $x, y, z \in B$ with $x + y = z$.

D. Spectral graph theory and pseudorandom graphs

ps4 D1. Let G be an n-vertex graph. The *Laplacian* of G is defined to be $L_G = D_G - A_G$, where A_G is the adjacency matrix of G and D_G a diagonal matrix whose entry corresponding to the vertex $v \in V(G)$ is the degree of v in G (so that L_G is a symmetric matrix with all row sums zero). Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of L_G, with $\lambda_1 = 0$ corresponding to the all-1 vector. Prove that for every $S \subset V(G)$ with $|S| \le n/2$, one has (writing $\overline{S} := V(G) \setminus S$)

$$e(S, \overline{S}) \ge \frac{1}{2}\lambda_2 |S|$$

ps4$\star$ D2. Let p be an odd prime and $A, B \subset \mathbb{Z}/p\mathbb{Z}$. Show that

$$\sum_{a \in A} \sum_{b \in B} \left(\frac{a+b}{p}\right) \le p\sqrt{p}$$

where (a/p) is the Legendre symbol defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod p \\ 1 & \text{if } a \text{ is a nonzero quadratic residue mod } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue mod } p \end{cases}$$

D3. *Quasirandom transitive graphs.* Prove that if an n-vertex d-regular vertex-transitive graph G satisfies

$$e(X, Y) - \tfrac{d}{n}|X||Y| \le \epsilon dn \qquad \text{for all } X, Y \subseteq V(G),$$

then all the eigenvalues of the adjacency matrix of G, other than the largest one, are at most $8\epsilon d$ in absolute value.

ps4 D4. Prove that the diameter of an (n, d, λ)–graph is at most $\lceil \log n / \log(d/\lambda) \rceil$. (The *diameter* of a graph is the maximum distance between a pair of vertices.)

D5. Let G be an n-vertex d-regular graph. Suppose n is divisible by k. Color the vertices of G with k colors (not necessarily a proper coloring) such that each color appears exactly n/k times. Suppose that all eigenvalues, except the top one, of the adjacency matrix of G are at most d/k in absolute value. Show that there is a vertex of G whose neighborhood contains all k colors.

ps4 D6. Prove that for every positive integer d and real $\epsilon > 0$, there is some constant $c > 0$ so that if G is an n-vertex d-regular graph with adjacency matrix A_G, then at least cn of the eigenvalues of A_G are greater than $2\sqrt{d-1} - \epsilon$.

ps4$\star$ D7. Show that for every d and r, there is some $\epsilon > 0$ such that if G is a d-regular graph, and $S \subset V(G)$ is such that every vertex of G is within distance r of S, then the top eigenvalue of the adjacency matrix of $G - S$ (i.e., remove S and its incident edges from G) is at most $d - \epsilon$.

ps4$\star$ D8. Prove or disprove: there exists an absolute constant C such that the adjacency matrix of every n-vertex Cayley graph has an eigenbasis in $\mathbb{C}^n$ (consisting of n orthonormal unit eigenvectors) all of whose coordinates are each at most $C/\sqrt{n}$ in absolute value.

E. Graph limits and homomorphism density inequalities

Note: A "graphon" is a symmetric measurable function $W\colon [0,1]^2 \to [0,1]$.

ps5

E1. *Weak regularity decomposition (instead of partition).*

(a) Let $\epsilon > 0$. Show that for every graphon W, there exist measurable $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0,1]$ and reals $a_1, \dots, a_k \in \mathbb{R}$, with $k < \epsilon^{-2}$, such that

$$\left\| W - \sum_{i=1}^{k} a_i \mathbf{1}_{S_i \times T_i} \right\|_{\square} \le \epsilon.$$

The above conclusion allows one to approximate an arbitrary graph(on) as a sum of most ϵ^{-2} components. In the next following parts, you will show how to recover a regularity partition from the approximation above.

(b) Show that the stepping operator $\mathcal{P}$ is contractive with respect to the cut norm, in the sense that if $W\colon [0,1]^2 \to \mathbb{R}$ is a measurable symmetric function, then $\|W_{\mathcal{P}}\|_\square \le \|W\|_\square$.

(c) Let $\mathcal{P}$ be a partition of $[0,1]$ into measurable sets. Let U be a graphon that is constant on $S \times T$ for each $S, T \in \mathcal{P}$. Show that for every graphon W, one has

$$\|W - W_{\mathcal{P}}\|_\square \le 2\|W - U\|_\square.$$

(d) Use (a) and (c) to give a different proof of the weak regularity lemma (with slightly worse bounds than the one given in class): show that for every $\epsilon > 0$ and every graphon W, there exists partition $\mathcal{P}$ of $[0,1]$ into $2^{O(1/\epsilon^2)}$ measurable sets such that $\|W - W_{\mathcal{P}}\|_\square \le \epsilon$.

E2. Define $W\colon [0,1]^2 \to \mathbb{R}$ by $W(x,y) = 2\cos(2\pi(x-y))$. Let G be a graph. Show that $t(G,W)$ is the number of ways to orient all edges of G so that every vertex has the same number of incoming edges as outgoing edges.

E3. Show that for every $\epsilon > 0$ there is some $C > 0$ such that if W is a graphon, and $S \subset [0,1]$ is a set of such that, writing $W \circ W(x,z) = \int_{[0,1]} W(x,y)W(y,z)\,dy$,

$$\int_{[0,1]} |W \circ W(s,z) - W \circ W(t,z)|\; dz > \epsilon$$

for all distinct $s, t \in S$, then $|S| \le C$.

E4. Let W be a $\{0,1\}$-valued graphon. Suppose graphons W_n satisfy $\|W_n - W\|_\square \to 0$ as $n \to \infty$. Show that $\|W_n - W\|_1 \to 0$ as $n \to \infty$.

ps5

E5. *"Regularity lemma" for bounded degree graphs.* The *r-local sample* of a graph G is defined to be the random rooted graph induced by all vertices within distance r from a uniform random vertex v of G, and setting v to be the root.

Show that for every $\epsilon > 0$ and $r, \Delta \in \mathbb{N}$ there exists $M = M(\epsilon, r, \Delta)$ such that if G is a graph with maximum degree at most Δ, then there exists a graph H on at most M vertices such that the r-local samples of G and H differ by at most ϵ in total variation distance.

E6. *Strong regularity lemma.* In this problem, you will give an alternate proof of the strong regularity lemma with explicit bounds.

Let $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \dots)$ be a sequence of positive reals. By repeatedly applying the weak regularity lemma, show that there is some $M = M(\boldsymbol{\epsilon})$ such that for every graphon W, there is a pair of partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[0,1]$ into measurable sets, such that $\mathcal{Q}$ refines $\mathcal{P}$, $|\mathcal{Q}| \le M$ (here $|\mathcal{Q}|$ denotes the number of parts of $\mathcal{Q}$),

$$\|W - W_{\mathcal{Q}}\|_\square \le \epsilon_{|\mathcal{P}|} \qquad \text{and} \qquad \|W_{\mathcal{Q}}\|_2^2 \le \|W_{\mathcal{P}}\|_2^2 + \epsilon_1^2.$$

Furthermore, deduce the strong regularity lemma in the following form: one can write

$$W = W_{\mathrm{str}} + W_{\mathrm{psr}} + W_{\mathrm{sml}}$$

where W_{str} is a k-step-graphon with $k \le M$, $\|W_{\mathrm{psr}}\|_\square \le \epsilon_k$, and $\|W_{\mathrm{sml}}\|_1 \le \epsilon_1$. State your bounds on M explicitly in terms of $\boldsymbol{\epsilon}$. (Note: the parameter choice $\epsilon_k = \epsilon/k^2$ roughly corresponds to Szemerédi's regularity lemma, in which case your bound on M should be an exponential tower of 2's of height $\epsilon^{-O(1)}$; if not then you are doing something wrong.)

ps5 E7. *Inverse counting lemma.* Using the moments lemma ($t(F,U) = t(F,W)$ for all F implies $\delta_\square(U,W) = 0$) and compactness of the space of graphons, deduce that for every $\epsilon > 0$, there exist $k \in \mathbb{N}$ and $\eta > 0$ such that if U and W are graphons such that $|t(F,U) - t(F,W)| \le \eta$ for all graphs F on k vertices, then $\delta_\square(U,W) \le \epsilon$.

ps5⋆ E8. *Generalized maximum cut.* For symmetric measurable functions $W, U \colon [0,1]^2 \to \mathbb{R}$, define

$$\mathcal{C}(W,U) := \sup_{\varphi} \langle W, U^\varphi \rangle = \sup_{\varphi} \int W(x,y) U(\varphi(x), \varphi(y))\, dx dy,$$

where φ ranges over all measure-preserving bijections on $[0,1]$. Extend the definition of $\mathcal{C}(\cdot,\cdot)$ to graphs by $\mathcal{C}(G,\cdot) := \mathcal{C}(W_G,\cdot)$, etc.

(a) Is $\mathcal{C}(U,W)$ continuous jointly in (U,W) with respect to the cut norm? Is it continuous in U if W is held fixed?

(b) Show that if W_1 and W_2 are graphons such that $\mathcal{C}(W_1,U) = \mathcal{C}(W_2,U)$ for all graphons U, then $\delta_\square(W_1,W_2) = 0$.

(c) Let $G_1, G_2, \ldots$ be a sequence of graphs such that $\mathcal{C}(G_n,U)$ converges as $n \to \infty$ for every graphon U. Show that $G_1, G_2, \ldots$ is convergent.

(d) Can the hypothesis in (c) be replaced by "$\mathcal{C}(G_n,H)$ converges as $n \to \infty$ for every graph H"?

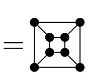
E9. (a) Let G_1 and G_2 be two graphs such that $\hom(F,G_1) = \hom(F,G_2)$ for every graph F. Show that G_1 and G_2 are isomorphic.

(b) Let G_1 and G_2 be two graphs such that $\hom(G_1,H) = \hom(G_2,H)$ for every graph H. Show that G_1 and G_2 are isomorphic.

E10. Fix $0 < p < 1$. Let G be a graph on n vertices with average degree at least pn. Prove:

(a) The number of labeled copies of $K_{3,3}$ in G is at least $(p^9 - o(1))n^6$.

ps5 (b) The number of labeled 6-cycles in G is at least $(p^6 - o(1))n^6$. (You may not use part (d) for part (b))

ps5 (c) The number of labeled copies of $Q_3 =$  in G is at least $(p^{12} - o(1))n^8$.

ps5⋆ (d) The number of labeled paths on 4 vertices in G is at least $(p^3 - o(1))n^4$.

ps5⋆ E11. Let $\mathcal{F}_m$ denote the set of all m-edge graphs without isolated vertices (up to isomorphism). Suppose $p \in [0,1]$ is a constant, and G_n is a sequence of graphs such that

$$\lim_{n\to\infty} \sum_{F \in \mathcal{F}_m} t(F, G_n) = \sum_{F \in \mathcal{F}_m} p^{|E(F)|}$$

for every positive integer m. Prove that G_n converges to the constant graphon p.

ps5⋆ E12. Prove there is a function $f\colon [0,1] \to [0,1]$ with $f(x) \ge x^2$ and $\lim_{x\to 0} f(x)/x^2 = \infty$ such that

$$t(K_4^-, W) \ge f(t(K_3, W))$$

for all graphons W. Here K_4^- is K_4 with one edge removed.

F. Fourier analysis and linear patterns

Some conventions: for $f\colon \mathbb{F}_p^n \to \mathbb{C}$ with prime p,

- $\widehat{f}(r) = \mathbb{E}_{x\in\mathbb{F}_p^n} f(x)\omega^{-r\cdot x}$ where $\omega = e^{2\pi i/p}$
- $\|f\|_s := (\mathbb{E}[|f|^s])^{1/s}$
- $\|\widehat{f}\|_\infty = \max_{r\in\mathbb{F}_p^n} |\widehat{f}(r)|$

ps5 F1. *Fourier does not control 4-AP counts.* Let $A = \{x \in \mathbb{F}_5^n : x\cdot x = 0\}$. Write $N = 5^n$.
  (a) Show that $|A| = (1/5 + o(1))N$ and $|\widehat{1_A}(r)| = o(1)$ for all $r \neq 0$.
  (b) Show that $|\{(x,y) \in \mathbb{F}_5^n : x, x+y, x+2y, x+3y \in A\}| \neq (5^{-4} + o(1))N^2$.

ps6 F2. *Linearity testing.* Show that for every prime p and real $\epsilon > 0$, there exists $\delta > 0$ such that if $f\colon \mathbb{F}_p^n \to \mathbb{F}_p$ is a function such that

$$\mathbb{P}_{x,y\in\mathbb{F}_p^n}(f(x) + f(y) = f(x+y)) \ge 1 - \delta$$

then there exists some $a \in \mathbb{F}_p^n$ such that

$$\mathbb{P}_{x\in\mathbb{F}_p^n}(f(x) = a_1x_1 + \cdots + a_nx_n) \ge 1 - \epsilon,$$

where in the above $\mathbb{P}$ expressions x and y are chosen i.i.d. uniform from $\mathbb{F}_p^n$.

ps6 F3. *Counting solutions to a single linear equation.*
  (a) Given a function $f\colon \mathbb{Z} \to \mathbb{C}$ with finite support, define $\widehat{f}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by

$$\widehat{f}(t) = \sum_{n\in\mathbb{Z}} f(n)e^{-2\pi i n t}.$$

  Let $c_1, \dots, c_k \in \mathbb{Z}$. Let $A \subset \mathbb{Z}$ be a finite set. Show that

$$|\{(a_1,\dots,a_k) \in A^k : c_1a_1 + \cdots + c_ka_k = 0\}| = \int_0^1 \widehat{1_A}(c_1t)\widehat{1_A}(c_2t)\cdots\widehat{1_A}(c_kt)\,dt.$$

  (b) Show that if a finite set A of integers contains $\beta|A|^2$ solutions $(a,b,c) \in A^3$ to $a+2b = 3c$, then it contains at least $\beta^2|A|^3$ solutions $(a,b,c,d) \in A^4$ to $a+b = c+d$.

ps6 F4. Let $a_1,\dots,a_m,b_1,\dots,b_m,c_1,\dots,c_m \in \mathbb{F}_2^n$. Suppose that the equation $a_i + b_j + c_k = 0$ holds if and only if $i = j = k$. Show that there is some constant $\epsilon > 0$ such that $m \le (2-\epsilon)^n$ for all sufficiently large n.

F5. *Strong arithmetic regularity lemma.* Show that for every $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \dots)$ with $1 \ge \epsilon_0 \ge \epsilon_1 \ge \cdots$ there exists $m = m(\boldsymbol{\epsilon})$ such that for every $f \colon \mathbb{F}_3^n \to [0,1]$ there exist a pair of subspaces $W \le U$ of $\mathbb{F}_3^n$ with codimW $\le m$ and a decomposition

$$f = f_{\mathrm{str}} + f_{\mathrm{psr}} + f_{\mathrm{sml}}$$

such that

- $f_{\mathrm{str}} = f_U$ and $f_{\mathrm{str}} + f_{\mathrm{sml}} = f_W$,
- $\|\widehat{f_{\mathrm{psr}}}\|_\infty \le \epsilon_{\operatorname{codim} U}$
- $\|f_{\mathrm{sml}}\|_2 \le \epsilon_0$

F6. *Counting lemma for 3-APs with restricted differences.* Let $f \colon \mathbb{F}_3^n \to [0,1]$ be written as $f = f_{\mathrm{str}} + f_{\mathrm{psr}} + f_{\mathrm{sml}}$ where

- f_{str} and $f_{\mathrm{str}} + f_{\mathrm{sml}}$ take values in $[0,1]$,
- $\|\widehat{f_{\mathrm{psr}}}\|_\infty \le \eta$, and
- $\|f_{\mathrm{sml}}\|_2 \le \epsilon$.

Let U be a subspace of $\mathbb{F}_3^n$. Show that there is some absolute constant C so that

$$\mathbb{E}_{x \in \mathbb{F}_3^n, y \in U}(f(x)f(x+y)f(x+2y) - f_{\mathrm{str}}(x)f_{\mathrm{str}}(x+y)f_{\mathrm{str}}(x+2y)\ ) \le C(|U^\perp|\eta + \epsilon)$$

F7. *Gowers U^2 uniformity norm.* Let Γ be a finite abelian group. For $f \colon \Gamma \to \mathbb{C}$, define

$$\|f\|_{U^2} := \left(\mathbb{E}_{x,h,h' \in \Gamma} f(x)\overline{f(x+h)f(x+h')}f(x+h+h')\right)^{1/4}.$$

(a) Show that the expectation above is always a nonnegative real number, so that the above expression is well defined. Also, show that $\|f\|_{U^2} \ge |\mathbb{E} f|$.

(b) For $f_1, f_2, f_3, f_4 \colon \Gamma \to \mathbb{C}$, let

$$\langle f_1, f_2, f_3, f_4 \rangle = \mathbb{E}_{x,h,h' \in \Gamma} f_1(x)\overline{f_2(x+h)f_3(x+h')}f_4(x+h+h').$$

Prove that

$$|\langle f_1, f_2, f_3, f_4 \rangle| \le \|f_1\|_{U^2} \|f_2\|_{U^2} \|f_3\|_{U^2} \|f_4\|_{U^2}$$

(c) By noting that $\langle f_1, f_2, f_3, f_4 \rangle$ is multilinear, and using part (b), show that

$$\|f+g\|_{U^2} \le \|f\|_{U^2} + \|g\|_{U^2}.$$

Conclude that $\|\ \|_{U^2}$ is a norm.

(d) Show that $\|f\|_{U^2} = \|\widehat{f}\|_{\ell^4}$, i.e.,

$$\|f\|_{U^2}^4 = \sum_{\gamma \in \widehat{\Gamma}} |\widehat{f}(\gamma)|^4.$$

Furthermore, deduce that if $\|f\|_\infty \le 1$, then

$$\|\widehat{f}\|_\infty \le \|f\|_{U^2} \le \|\widehat{f}\|_\infty^{1/2}.$$

(This gives a so-called "inverse theorem" for the U^2 norm: if $\|f\|_{U^2} \ge \delta$ then $|f(\gamma)| \ge \delta^2$ for some $\gamma \in \widehat{\Gamma}$, i.e., if f is not U^2-uniform, then it must correlate with some character.)

G. Structure of set addition

ps6 G1. Show that for every real $K \geq 1$ there is some C_K such that for every finite set A of an abelian group with $|A+A| \leq K|A|$, one as $|nA| \leq n^{C_K}|A|$ for every positive integer n.

ps6⋆ G2. Show that there is some constant C so that if S is a finite subset of an abelian group, and k is a positive integer, then $|2kS| \leq C^{|S|}\,|kS|$.

ps6⋆ G3. Show that for every sufficiently large K there is there some finite set $A \subset \mathbb{Z}$ such that $|A+A| \leq K|A|$ and $|A-A| \geq K^{1.99}|A|$.

ps6⋆ G4. Show that for every finite subsets A, B, C in an abelian group, one has

$$|A+B+C|^2 \leq |A+B|\,|A+C|\,|B+C|\,.$$

ps6 G5. Let $A \subset \mathbb{Z}$ with $|A| = n$.

(a) Let p be a prime. Show that there is some integer t relatively prime to p such that $\|at/p\|_{\mathbb{R}/\mathbb{Z}} \leq p^{-1/n}$ for all $a \in A$.

(b) Show that A is Freiman 2-isomorphic to a subset of $[N]$ for some $N = (4+o(1))^n$.

(c) Show that (b) cannot be improved to $N = 2^{n-2}$.

(You may use the fact that the smallest prime larger than m has size $m + o(m)$.)

G6. Let $r_3(N)$ denote the size of the largest 3-AP-free subset of $[N]$. Show that there is some constant $c > 0$ so that if A is 3-AP-free, then $|A+A| \geq c|A|^{1+c} r_3(|A|)^{-c}$.

ps6 G7. Let $A \subset \mathbb{F}_2^n$ with $|A| = \alpha 2^n$.

(a) Show that if $|A+A| < 0.99 \cdot 2^n$, then there is some $r \in \mathbb{F}_2^n \setminus \{0\}$ such that $|\widehat{1_A}(r)| > c\alpha^{3/2}$ for some constant $c > 0$.

(b) By iterating (a), show that $A+A$ contains 99% of a subspace of codimension $O(\alpha^{-1/2})$.

(c) Deduce that $4A$ contains a subspace of codimension $O(\alpha^{-1/2})$ (i.e., Bogolyubov's lemma with better bounds than the one shown in class)

G8. Prove that there is some $C > 0$ so that every set of n integers has a 3-AP-free subset of size $ne^{-C\sqrt{\log n}}$.

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