Lecture 18

April 22nd, 2004

Embedding Theorems for Sobolev spaces

Sobolev Embedding Theorem. *Let Ω a bounded domain in $\mathbb{R}^n$, and $1 \leq p < \infty$.*

$$W_0^{1,p}(\Omega) \subseteq \begin{cases} L^{\frac{np}{n-p}}(\Omega), & p < n \\ \mathcal{C}^{0,\alpha}(\Omega), \alpha = 1 - \frac{n}{p}, & p > n, \\ \qquad \textit{i.e in particular} \ \subseteq \ \mathcal{C}^0(\Omega). & \end{cases}$$

Furthermore, those embeddings are continuous in the following sense: there exists $C(n, p, \Omega)$ such that for $u \in W_0^{1,p}(\Omega)$

$$||u||_{L^{\frac{np}{n-p}}(\Omega)} \leq C \cdot ||\nabla u||_{L^p(\Omega)}, \quad \forall p < n$$

$$\sup_{\Omega} |u| \leq C' \cdot Vol(\Omega)^{\frac{1}{n} - \frac{1}{p}} \cdot ||Du||_{L^p(\Omega)}, \quad \forall p > n.$$

We start with a function whose derivative and itself belong to L^p. The above theorem gives us more regularity for the function – it belongs to $L^{p \cdot \frac{n}{n-p}}$ – based on its regular derivative.

Proof. $\mathcal{C}_0^1(\Omega)$ is dense in $W_0^{1,p}(\Omega)$. We prove first for $u \in \mathcal{C}_0^1(\Omega)$ and will later justify why the proof actually extends to the larger space.

Case $p = 1$. fix an index $i \in \{1, \ldots, n\}$ and observe

$$u(x) = \int_{-\infty}^{x_i} \mathrm{D}_i u(x_1, \ldots, t, \ldots, x_n) dt.$$

From which

$$
\begin{aligned}
|u(x)| &\le \int_{-\infty}^{x_i} |\mathrm{D}_i u|(x_1,\ldots,t,\ldots,x_n)dt \\
&\le \int_{-\infty}^{\infty} |\mathrm{D}_i u|(x_1,\ldots,t,\ldots,x_n)dt. \qquad (1)
\end{aligned}
$$

Write this down for each i, take a product of the terms and take the $n-1^{\text{th}}$ root of the result to yield altogether

$$
|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} |\mathrm{D}_i u| dx_i \Big)^{\frac{1}{n-1}}.
$$

Quick Reminder. Hölder's inequality (HI) tells us

$$
\frac{1}{p} + \frac{1}{q} = 1 \quad \Rightarrow \quad \int u \cdot v \le \Big(\int u^p \Big)^{\frac{1}{p}} \cdot \Big(\int v^p \Big)^{\frac{1}{q}},
$$

or more generally

$$
\frac{1}{p_1} + \ldots + \frac{1}{p_k} = 1 \quad \Rightarrow \quad \int u_1 \cdots u_k \le \Big(\int u_1^{p_1} \Big)^{\frac{1}{p_1}} \cdots \Big(\int u_k^{p_k} \Big)^{\frac{1}{p_k}}.
$$

Coming back to our inequality, we integrate over the x_1 axis and subsequently apply the Hölder inequality with $k = n-1,\ p_i = n-1$ –

$$
\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} |\mathrm{D}_i u| dx_i \Big)^{\frac{1}{n-1}} dx_1. \\
&= \Big(\int_{-\infty}^{\infty} |\mathrm{D}_1 u| dx_1 \Big)^{\frac{1}{n-1}} \cdot \int_{-\infty}^{\infty} \prod_{i=2}^{n} \Big(\int_{-\infty}^{\infty} |\mathrm{D}_i u| dx_i \Big)^{\frac{1}{n-1}} dx_1. \\
&\underset{\text{Hölder's Ineq.}}{\le} \Big(\int_{-\infty}^{\infty} |\mathrm{D}_1 u| dx_1 \Big)^{\frac{1}{n-1}} \cdot \prod_{i=2}^{n} \Big(\int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} |\mathrm{D}_i u| dx_i \Big]^{\frac{n-1}{n-1}} dx_1 \Big)^{\frac{1}{n-1}}. \\
&= \Big(\int_{-\infty}^{\infty} |\mathrm{D}_1 u| dx_1 \Big)^{\frac{1}{n-1}} \cdot \Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\mathrm{D}_2 u| dx_2 dx_1 \Big)^{\frac{1}{n-1}}. \\
&\qquad \cdot \prod_{i=3}^{n} \Big[\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} |\mathrm{D}_i u| dx_i \Big) dx_1 \Big]^{\frac{1}{n-1}}.
\end{aligned}
$$

Now courageously continuing with this confusing calculation, we integrate over the x_2 axis. This is the reason we singled out the second terms from the $n-2$ others ones; if we would have integrated now over the x_j axis we would have choosen a term involving integration over that axis. And indeed now the middle term is a constant WRT this operation, that is only the other two terms appear in this integral, hence –

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}dx_1dx_2$$

$$\leq \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathrm{D}_2u|dx_2dx_1\Big)^{\frac{1}{n-1}}\cdot\bigg(\int_{-\infty}^{\infty}\Big\{\Big(\int_{-\infty}^{\infty}|\mathrm{D}_1u|dx_1\Big)^{\frac{1}{n-1}}\cdot\prod_{i=3}^{n}\Big[\int_{-\infty}^{\infty}|\mathrm{D}_iu|dx_i\Big]dx_2\Big\}\bigg)^{\frac{1}{n-1}}.$$

and using the Hölder Inequality the second term transforms, and we have

$$= \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathrm{D}_2u|dx_2dx_1\Big)^{\frac{1}{n-1}}\cdot\Big(\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}|\mathrm{D}_1u|dx_1\Big]^{\frac{n-1}{n-1}}dx_2\Big)^{\frac{1}{n-1}}\cdot$$

$$\cdot\Big(\prod_{i=3}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathrm{D}_iu|dx_idx_1dx_2\Big)^{\frac{1}{n-1}}.$$

In the same vein, we now isolate among the n terms the only term involving integration over the x_3 axis, integrate over that axis and then once again apply the Hölder Inequality for the remaining $n-1$ terms (at each stage we always have $n-1$ terms except from the isolated one; the Hölder Inequality allows us to lift the $\frac{1}{n-1}$ exponent and let another new dx_i come in to the integral of those $n-1$ terms).

Finally, therefore, we will arrive at

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}dx_1\cdots dx_n \leq \prod_{j=1}^{n}\Big(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\mathrm{D}_ju|dx_1\cdots dx_n\Big)^{\frac{1}{n-1}}.$$

In other words if we restrict to Ω

$$\Big(||u||_{L^{\frac{n}{n-1}}(\Omega)}\Big)^{\frac{n}{n-1}} \leq \prod_{j=1}^{n}\Big(\int_{\Omega}|\mathrm{D}_ju|d\mathbf{x}\Big)^{\frac{1}{n-1}}.$$

or still

$$||u||_{L^{\frac{n}{n-1}}(\Omega)} \leq \prod_{j=1}^{n} \Big(\int_\Omega |\mathrm{D}_j u| d\mathbf{x} \Big)^{\frac{1}{n}} \leq \frac{1}{n} \cdot \sum_{j=1}^{n} \cdot \int_\Omega |\mathrm{D}_j u| d\mathbf{x} \leq \frac{1}{n} \cdot \sum_{j=1}^{n} \cdot \int_\Omega |\mathrm{D} u| d\mathbf{x} = \int_\Omega |\mathrm{D} u| d\mathbf{x}$$

$$= ||\nabla u||_{L^1(\Omega)}.$$

This concludes the $p = 1 < n$ case. Let us remark that of course we neglected at the last steps to seek the *best possible Sobolev constant* and contented ourselves with the constant 1:

$$||u||_{L^{\frac{n \cdot 1}{n-1}}(\Omega)} \leq 1 \cdot ||\nabla u||_{L^1(\Omega)}.$$

In fact the best possible Sobolev constant c is achieved for $\Omega = B(0, r)$, $u = \mathbb{I}_{B(0,r)}$ ($\mathbb{I}_A$ is the *characteristic function on the set* A, evaluating to 1 on A and 0 otherwise); believing that, we compute

$$\mathrm{Vol}(B(0, r))^{\frac{n}{n-1}} = c \cdot \int_{B(0,r)} |D\mathbb{I}_{B(0,r)}| d\mathbf{x} = c \cdot \int_{\overline{B(0,r)}} |\delta_{\partial B(0,r)}| d\mathbf{x} = c \cdot \mathrm{Area}(S(r)),$$

i.e

$$(\omega_n r^n)^{\frac{n}{n-1}} = c \cdot n \omega_n r^{n-1} \quad \Rightarrow \quad c = \frac{1}{n \sqrt[n]{\omega_n}}.$$

Case $1 < p < \infty$. A little trick will make our previous work apply to this case as well. Let $\gamma > 1$ be a constant to be specified. We have by our previous case

$$||\, |u|^\gamma ||_{L^{\frac{n}{n-1}}(\Omega)} \leq \int_\Omega \big| \mathrm{D} |u|^\gamma \big| d\mathbf{x} \leq \gamma \int_\Omega |u|^{\gamma - 1} \cdot |\mathrm{D} u| d\mathbf{x}.$$

Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. One has using the Hölder Inequality

$$\Big(\int_\Omega |u|^{\gamma \cdot \frac{n}{n-1}} d\mathbf{x} \Big)^{\frac{n}{n-1}} \leq \Big(\int_\Omega |u|^{(\gamma-1)q} d\mathbf{x} \Big)^{\frac{1}{q}} \cdot \Big(\int_\Omega |\mathrm{D} u|^p d\mathbf{x} \Big)^{\frac{1}{p}}.$$

We have $q = \frac{p}{p-1}$. Choose $\gamma = \frac{n-1}{n-p} \cdot p$ in order to have $(\gamma - 1) q = \frac{n}{n-1} \cdot \gamma$. Hence

$$\Big(\int_\Omega |u|^{(\frac{n-1}{n-p}\cdot p)\cdot \frac{n}{n-1}} d\mathbf{x} \Big)^{\frac{n}{n-1}} \leq \Big(\int_\Omega |u|^{(\frac{n-1}{n-p}\cdot p)-1)\cdot (\frac{p}{p-1})} d\mathbf{x} \Big)^{\frac{p-1}{q}} \cdot \Big(\int_\Omega |\mathrm{D}u|^p d\mathbf{x} \Big)^{\frac{1}{p}},$$

or succintly

$$||u||_{L^{\frac{np}{n-p}}(\Omega)} = \Big\{ \int_\Omega |u|^{\frac{np}{n-p}} \Big\}^{\frac{n-1}{n} - \frac{p-1}{p}} \leq \frac{n-1}{n-p} \cdot p||\nabla u||_{L^p(\Omega)}.$$

This deals with the case $p < n$ indeed. We remark that characteristic functions no longer give the best Sobolev constants in the case $1 < p < n$.

Remark. The above proof holds and is valid for $u \in \mathcal{C}_0^1(\Omega)$! We did not prove for distributional coefficient. If u is only in $W_0^{1,p}(\Omega)$, take a sequence $\{u_m\} \subseteq \mathcal{C}_0^1(\Omega)$ such that $u_m \to u$ in the $W_0^{1,p}(\Omega)$-norm. This means that also

$$||u_i - u_j||_{L^{\frac{np}{n-p}}(\Omega)} \leq c \cdot ||\mathrm{D}u_i - du_j||_{L^p(\Omega)} \to 0.$$

$\{u_m\}$ is thus a Cauchy sequence in $L^{\frac{np}{n-p}}(\Omega)$. $L^{\frac{np}{n-p}}(\Omega)$ is a Banach space d'aprês Riesz-Fischer, i.e $u' := \lim\{u_m\} \in L^{\frac{np}{n-p}}(\Omega)$; $\Leftrightarrow$ $u' = u$ is in that space too. Now

$$\begin{array}{ccc} ||u_m||_{L^{\frac{np}{n-p}}(\Omega)} & \leq c \cdot & ||\mathrm{D}u_m||_{L^p(\Omega)} \to 0. \\ \downarrow & & \downarrow \\ ||u||_{L^{\frac{np}{n-p}}(\Omega)} & \leq c \cdot & ||\mathrm{D}u||_{L^p(\Omega)} \to 0. \end{array}$$

So we our Theorem applies equally well to functions in the larger space. In this last line we needed also mention that $\mathrm{D}u_m \to \mathrm{D}u$, but this is true since $\{\mathrm{D}u_m\}$ is a Cauchy sequence from same computation as above. Its limit lies in L^p again as this is a Banach space and so indeed $\mathrm{D}u_m \to \mathrm{D}u$ in L^p and hence also $||\mathrm{D}u_m||_{L^p(\Omega)} \to ||\mathrm{D}u||_{L^p(\Omega)}$.

Case $p > n$. We postpone the proof of this case to state a Corollary. ■

Corollary. By iterating, $\forall k \geq 2$ holds

$$W_0^{k,p}(\Omega) \subseteq \begin{cases} L^{\frac{np}{n-k\cdot p}}(\Omega), & kp < n \\ \mathcal{C}^m, & 0 \leq m \leq k - \frac{n}{p}. \end{cases}$$

Proof. For instance, if $k = 2$, $u \in W^{2,p} \quad \Rightarrow \quad u, \mathrm{D}u \in W^{1,p}$. By the $k = 1$ case above we have $u, \mathrm{D}u \in L^{\frac{np}{n-p}}$. That means $\mathrm{D}u \in W^{1,\frac{np}{n-p}} \quad \Rightarrow \quad$ (by $k = 1$ case once again) $u \in W^{1,p'}$ where

$$p' = \frac{n \cdot (\frac{np}{n-p})}{n - (\frac{np}{n-p})} = \frac{n^2 p}{n^2 - np - np} = \frac{np}{n-2p}.$$

This proof repeated carries over $\forall k \in \mathbb{N}$.

Now for the second inclusion, the promised postponed. We will need the following lemma en passant.

Lemma. *Let Ω be a bounded domain, $B :=$Ball $\subseteq \Omega, u \in W^{1,1}$. Then for all $x \in \Omega$*

$$\left|u(x) - \frac{1}{Vol(B)} \int_B u d\mathbf{x}\right| \equiv \left|u(x) - \fint_B u d\mathbf{x}\right| \leq c \cdot \int_B \frac{|Du(y)|}{|x-y|^{n-1}} d\mathbf{y}.$$

Proof. By our density theorem $\mathcal{C}_0^1(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and thus work with u in the former. Take $x, y \in \Omega$. Let $\omega := \frac{y-x}{|y-x|}$,

$$u(x) - u(y) = \int_0^{|x-y|} \mathrm{D}_r u(x + r\omega) dr.$$

Integrating over some ball B

$$\mathrm{Vol}(B) \cdot u(x) - \int_B u(y) = \int_B \Big(\int_0^{|x-y|} \mathrm{D}_r u(x + r\omega) dr \Big) d\mathbf{y}.$$

Put

$$v(x) = \begin{cases} \mathrm{D}_r u(x), & x \in \Omega \\ 0, & x \notin \Omega. \end{cases}$$

Take now a particular ball $B(x,R) \subseteq \Omega$ to get

$$\big|u(x) - \fint_B u(y)d\mathbf{y}\big| \leq \frac{1}{\mathrm{Vol}(B)} \int_{|x-y|<2R} \big(\int_0^\infty |v(x+r\omega)|dr\big)d\mathbf{y}.$$

Switch order of integration, and change coordinates to spherical ones

$$= \frac{1}{\mathrm{Vol}(B)} \int_0^\infty \Big(\int_0^{2R} \big(\int_{S^{n-1}(1)} |v(x+r\omega)|\rho^{n-1} d\omega_{S^{n-1}(1)}\big)d\rho\Big)dr$$

after rescaling, where (ρ,ω) are the spherical coordinates, i.e ω are coordinates on the unit sphere.

Now

$$= \frac{(2R)^n}{n\mathrm{Vol}(B)} \int_0^\infty \big(\int_{S^{n-1}(1)} |v(x+r\omega)| d\omega_{S^{n-1}(1)}\big)dr$$

$$= \frac{(2R)^n}{n\mathrm{Vol}(B)} \int_0^\infty \big(\int_{S^{n-1}(1)} \frac{|v(x+r\omega)|}{r^{n-1}} r^{n-1} d\omega_{S^{n-1}(1)}\big)dr.$$

Set $z := x + r\omega, \;\; \rightarrow \;\; r = |r\omega| = |x-z|, \; r^{n-1}d\omega_{S^{n-1}(1)}dr = d\mathbf{z},$

$$= \frac{(2R)^n}{n\mathrm{Vol}(B)} \int_B \frac{|v(z)|}{|x-z|^{n-1}} d\mathbf{z},$$

and as $B(x,R) \subseteq \Omega \qquad \Rightarrow$

$$\leq \frac{(2R)^n}{n\mathrm{Vol}(B)} \int_\Omega \frac{|\mathrm{D}_r u(z)|}{|x-z|^{n-1}} d\mathbf{z}. \qquad \blacksquare$$

Claim.

$$\int_{B_R} |x-y|^{1-n} |\mathrm{D}u(y)| d\mathbf{y} \leq C R^{1-\frac{n}{p}} ||\mathrm{D}u||_{L^p(B_R)}, \qquad \forall p > n.$$

for $B_R := B(x_0, R) \subseteq \mathbb{R}^n$.

Proof. By the Hölder inequality, $\forall q$ such that $\frac{1}{q}+\frac{1}{p}=1$

$$\int_{B_R}|x-y|^{1-n}|\mathrm{D}u(y)|d\mathbf{y} \le \Big\{\int_{B_R}|x-y|^{(1-n)q}d\mathbf{y}\Big\}^{\frac{1}{q}}\cdot||\mathrm{D}u||_{L^p(B_R)}$$

$$\le \sup_{x\in\Omega}\Big\{\int_{B_R}|x-y|^{(1-n)q}d\mathbf{y}\Big\}^{\frac{1}{q}}\cdot||\mathrm{D}u||_{L^p(B_R)}$$

$$= c\cdot\Big\{\int_{B_R}|x_0-y|^{(1-n)q}d\mathbf{y}\Big\}^{\frac{1}{q}}\cdot||\mathrm{D}u||_{L^p(B_R)}$$

$$= c\cdot\Big\{\int_0^R r^{(1-n)q}r^{n-1}dr\Big\}^{\frac{1}{q}}\cdot||\mathrm{D}u||_{L^p(B_R)}$$

$$= c\cdot\Big\{\int_0^R r^{\frac{n-1}{1-p}}dr\Big\}^{\frac{1}{q}}\cdot||\mathrm{D}u||_{L^p(B_R)}$$

$$= c\cdot\big(\frac{n-1}{1-p}+1\big)R^{\left(\frac{n-1}{1-p}+1\right)/q}\cdot||\mathrm{D}u||_{L^p(B_R)}$$

$$= C(n,p)R^{\frac{p-n}{p}}\cdot||\mathrm{D}u||_{L^p(B_R)}$$

as $\big(\frac{n-1}{1-p}+1\big)\cdot\frac{1}{q}=\big(\frac{n-1}{1-p}+1\big)\cdot\frac{p}{1-p}=-\frac{n-1}{p}+\frac{p-1}{p}=\frac{p-n}{p}.$ ∎

Now we can finally, combining those last two results conclude the second inclusion in our Theorem as well as the estimate therein. First, using the triangle inequality together with the first lemma we have

$$|u(x)-u(y)|\le\big|u(x)-\fint_B ud\mathbf{x}\big|+\fint_B ud\mathbf{y}-u(y)\big|\le 2c\cdot\int_B\frac{|Du(y)|}{|x-y|^{n-1}}d\mathbf{y}$$

which in turn is

$$\le c(n,p)|x-y|^{1-\frac{n}{p}}||\mathrm{D}u||_{L^p(B)}$$

once we choose a ball $B=B(x,|x-y|)$ and apply the Claim. Since this is for any $x,y\in\Omega$, and $u\in W^{1,p}(\Omega)$ then $u\in\mathcal{C}^{1-\frac{n}{p}}(\Omega)$, if $p>n$.

Second and finally, we have as well

$$|u(x)| \leq \left|u(x) - \fint_B u d\mathbf{x}\right| \leq 2c \cdot \int_\Omega \frac{|Du(y)|}{|x-y|^{n-1}} d\mathbf{y} \leq c(n,p) \cdot \mathrm{diam}(\Omega)^{1-\frac{n}{p}} \ ||\mathrm{D}u||_{L^p(\Omega)}$$

$$= c'(n,p) \cdot \mathrm{Vol}(\Omega)^{\frac{1}{n}-\frac{1}{p}} \ ||\mathrm{D}u||_{L^p(\Omega)}$$

which gives the desired sup norm. Indeed for $k \geq 2$ the Corollary follows by iterating: we get first Hölder regularity of u, then we have $\mathrm{D}u$ is $W_0^{1,p}(\Omega)$ so we apply the first Theorem to it and get $\mathrm{D}u$ is Hölder and so on. ■