Lecture 14

April 6,th 2004

Extending interior Schauder estimates to flat boundary part

Theorem. *$u \in \mathcal{C}^{2,\alpha}(\Omega \cap T), Lu = f, u = 0$ on T, with $0 < \alpha < 1$. Assume coefficients are bounded in $\mathcal{C}^{2,\alpha}(\Omega \cap T)$ as well as uniformly elliptic. Then $\forall \Omega' \cap T' \Subset \Omega \cap T$, $\exists\, c = c(\Lambda, n, \Omega', \Omega, T', T)$ such that*

$$||u||_{C^{2,\alpha}(\Omega' \cap T')} \leq c\big(||u||_{C^0(\Omega \cap T)} + ||f||_{C^\alpha(\Omega \cap T)}\big).$$

Proof. As in the last remark we see that our proof consisted of perturbing the equation at any $x_0 \in \Omega'$ and relying on our constant coefficients estimates and interpolation methods. Both of these hold upto the flat boundary from our previous work. ■

Global Schauder estimates

Theorem. *Let Ω be a $\mathcal{C}^{2,\alpha}$ domain and $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})^\star$ with $0 < \alpha < 1$. Let L be uniformly elliptic with $\mathcal{C}^\alpha(\bar{\Omega})$ bounds on coefficients . Let*

$$\begin{aligned} Lu = f, &\quad f \in \mathcal{C}^\alpha(\bar{\Omega}), \\ u = \varphi &\quad \text{on } \partial\Omega. \end{aligned}$$

Then $\exists\, c = c(\Omega, \Lambda, n)$ such that

$$||u||_{C^{2,\alpha}(\Omega)} \leq c\big(||u||_{C^0(\Omega)} + ||f||_{C^\alpha(\Omega)} + ||\varphi||_{C^{2,\alpha}(\partial\Omega)}\big).$$

* We note that Gilbarg-Trudinger intend by this notation *locally* Hölder while we will take it henceforth to mean globally Hölder in the sense that we assume $\sup_{x_0 \neq y_0 \in \bar{\Omega}} \frac{\mathrm{D}^2 u(x_0) - \mathrm{D}^2 u(y_0)}{|x_0 - y_0|^\alpha}$ is finite.

Here we let $||\varphi||_{C^{2,\alpha}(\partial\Omega)} := \inf_{\tilde{\varphi}:\Omega\to\mathbb{R}} ||\tilde{\varphi}||_{C^{2,\alpha}(\Omega)}$.

Proof. It is enough to prove for the case of zero boundary values: if we can solve the Dirichlet problem

$$\begin{array}{rclcc} Lv & = & f - L\varphi =: f' \in \mathcal{C}^\alpha & \text{on} & \bar{\Omega}, \\ v & = & 0 & \text{on} & \partial\Omega. \end{array}$$

we can also solve our original one by setting $v + \varphi$ solves the original equation. And if we have the above announced estimates for v then by the triangle inequality (for the relevant *norms*) and the uniform ellipticity (which gives $||L\varphi||_{C^\alpha(\Omega)} \leq c \cdot ||\varphi||_{C^{2,\alpha}(\Omega)}$) the same estimates will hold for u, possibly with a different constant.

So indeed we may assume $\varphi = 0$.

By definition of a $\mathcal{C}^{2,\alpha}$ domain $\exists \Psi, \Psi^{-1} \in \mathcal{C}^{2,\alpha}(\mathbb{R}^n \to \mathbb{R}^n)$ mapping each small portion of the boundary of Ω, say $B(x_0, R) \cap \partial\Omega$ for $x_0 \in \partial\Omega$ to flat boundary. We set as in computations in the past $\tilde{u} := u \circ \Psi^{-1}$ and then $\mathrm{D}\tilde{u} = \mathrm{D}u \circ \Psi^{-1\prime}$, $\mathrm{D}^2\tilde{u} = \mathrm{D}^2 u \cdot \Psi^{-1\prime} + \mathrm{D}u \cdot \mathrm{D}^2\Psi^{-1}$. These computations convince us once more that the relevant norms on a, b, c and $\tilde{a}, \tilde{b}, \tilde{c}$ are equivalent using $\Psi, \Psi^{-1} \in \mathcal{C}^{2,\alpha}$ (e.g we find $||\tilde{b}||_{C^\alpha(\Omega)} \leq ||b||_{C^\alpha(\Omega)}(|\Psi|_{C^{1,\alpha}(+)}|\Psi|_{C^{2,\alpha}(\Omega)} \leq C \cdot \Lambda)$.

We have for the flat boundary

$$||\tilde{u}||_{C^{2,\alpha}(\Psi(B(x_0,\frac{1}{2}R)\cap\bar{\Omega}))} \leq c\big(||\tilde{u}||_{C^0(\Psi(B(x_0,R)\cap\bar{\Omega}))} + ||\tilde{f}||_{C^\alpha(\Psi(B(x_0,R)\cap\bar{\Omega}))}\big).$$

Now by our above work we know this holds also for u in $B(x_0, R) \cap \bar{\Omega}$

$$||u||_{C^{2,\alpha}(B(x_0,\frac{1}{2}R)\cap\bar{\Omega})} \leq c\big(||u||_{C^0(B(x_0,R)\cap\bar{\Omega})} + ||f||_{C^\alpha(B(x_0,R)\cap\bar{\Omega})}\big).$$

Now we patch up the estimates over a countable cover of $\partial\Omega$ by small balls $\{B(x_i, \frac{1}{2}R_i)\}$. $\partial\Omega$ being compact we may choose a finite subcover say after relabeling $\{B(x_i, \frac{1}{2}R_i)\}_{i=1}^N$. Finally we adjoin to these estimates an interior estimate for some Ω' such that $\Omega \setminus \cup_{i=1}^N B(x_i, \frac{1}{2}R_i) \Subset \Omega' \Subset \Omega$. And having this we are done by analysing the different cases that might arise in a similar fashion to previous proofs. ∎

Banach Spaces

$\mathcal{L}$et V be a vector space equipped with a norm $||\cdot|| : V \to \mathbb{R}$ i.e *i)* $||x|| \geq 0$ with equality $\Leftrightarrow x = 0$; *ii)* $||\alpha x|| = |\alpha| ||x||$; *iii)* $\Delta-$ inequality. With a norm we have a metric $d(x, y) := ||x-y||$ and we can talk about topology induced from it, convergence etc.

Cauchy sequence: $\{x_i\}$ such that $d(x_n, x_m) \xrightarrow{N\to\infty} 0, \ \ \forall m, n \geq N$.

Banach space: a normed space complete WRT the norm metric $\Leftrightarrow$ every Cauchy sequence converges (WRT the norm metric) in V (limit in V).

We mention in passing a few examples.

- The Bolzano-Weierstrass theorem showing $(\mathbb{R}^n, |\cdot|)$ is complete carries over to show finite dimensional normed spaces are Banach.

- $(\mathcal{C}^0(\Omega), ||\cdot||_{L^1})$ is incomplete, so is not Banach;

- On the other handwhile $(\mathcal{C}^0(\Omega), ||\cdot||_{C^0(\Omega)})$ and in general $(\mathcal{C}^{k,\alpha}(\Omega), ||\cdot||_{Ck,\alpha})$ are Banach, as can be demonstrated using the Arzelà-Ascoli theorem [cf. Peterson, *Riemannian Geometry*, Chapter 10].

- Sobolev spaces are yet another example.

$\mathcal{C}$ontraction Mapping Theorem. *Let $\mathcal{B}$ a Banach space and $T : \mathcal{B} \to \mathcal{B}$ a contraction mapping (*WRT *to the norm metric). Then T has a unique fixed point.*

Proof. Here the assumption translates into $||Tx - Ty|| \leq \theta \cdot ||x - y||$ for $\theta \in [0, 1)$. The idea is to look at the sequence $\{x_n := T^n x_0\}$ and show it is Cauchy using the Δ-inquality . Let $x \in V$ be its limit; we see that

$$Tx = T \lim x_n = \lim Tx_n \text{ (by continuity of T!) } = \lim x_{n+1} = x.$$

As for uniqueness, if x, y are two fixed points,

$$||x-y|| = ||Tx-Ty|| \leq \theta||x-y|| \;\Rightarrow\; ||x-y|| = 0$$

and by the norm properties $x = y$. ■