Now we would like to extend our estimates to general domains.

Proposition.   *Let Ω be a $\mathcal{C}^{2,\alpha}$ domain and $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$, with $0 < \alpha < 1$. Given $\epsilon > 0, \exists\, c = c(\epsilon, \Omega)$*

s.t. for $k = 0, 1, \beta \in (0,1)$ and $k = 2, \beta < \alpha$

$$||u||_{C^{k,\beta}(\Omega)} \leq c \cdot ||u||_{C^0(\Omega)} + \epsilon \cdot |u|_{C^{2,\alpha}(\Omega)}.$$

Note this is a global estimate, i.e. upto the boundary.

We will use the remark from last time concerning domains with a portion of a hyperplane on the boundary which will provide us with a needed estimate. We choose a $\mathcal{C}^{2,\alpha}$ injective function Ψ which maps $B(x,r) \cap \Omega$ $(x \in \partial\Omega)$ in such a manner as to map $B(x,r) \cap \partial\Omega$ onto a portion of a hyperplane. Its inverse $S := \Psi^{-1}$ is also $\mathcal{C}^{2,\alpha}$. Set $\tilde{u} := u \circ \Psi^{-1}$, the pulled-back function, and $T' := \Psi(B_{r/2}(x) \cap \partial\Omega)$. For the domain in the image we have estimates as just mentioned

$$|\tilde{u}|_{C^{k,\beta}(\Psi(B_{r/2}(x)) \cap \tilde{T})} \leq c(\epsilon) \cdot |\tilde{u}|_{C^0(\Psi(B_r(x)))} + \epsilon \cdot |\tilde{u}|_{C^{2,\alpha}(\Psi(B_{r/2}(x)) \cap \tilde{T})}. \tag{1}$$

Let $S = (S_{(1)}, \ldots, S_{(n)})$. Now calculate

$$D_i \tilde{u} = D_l u \cdot (S_{(l)})_i \equiv D_l u \cdot D_i(S_{(l)}) \text{ summation over } l \text{ understood}$$

$$D_j D_i \tilde{u} = D_k D_l u (S_{(k)})_j (S_{(l)})_i + D_l u D_j (S_{(l)})_i$$

$$|u|_{C^0(B_r(x))} = |\tilde{u}|_{C^0(\Psi(B_r(x)))}.$$

Now

$$|\tilde{u}|_{C^\alpha(\Psi(B_r(x)))} = \sup_{y_1 \neq y_2} \frac{|\tilde{u}(y_1) - \tilde{u}(y_2)|}{|y_1 - y_2|^\alpha} = \sup_{x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|\Psi(x_1) - \Psi(x_2)|^\alpha}$$

$$\leq \sup_{x_1 \neq x_2} |u|_{C^\alpha(B_r(x))} \frac{|x_1 - x_2|^\alpha}{|\Psi(x_1) - \Psi(x_2)|^\alpha}.$$

Now since Ψ, Ψ^{-1} are $\mathcal{C}^{2,\alpha}$ they are in particular Lipschitz ($\mathcal{C}^{0,1}$), i.e $\exists K > 0$ s.t.

$$|\Psi(x_1) - \Psi(x_2)| \leq K|x_1 - x_2|, \quad |\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K|y_1 - y_2|,$$

or

$$1/K \cdot |x_1 - x_2| \leq |\Psi(x_1) - \Psi(x_2)| \leq K \cdot |x_1 - x_2|.$$

When plugged-in to our previous computation this yields

$$\leq |u|_{C^\alpha(B_r(x))} \cdot K^{-\alpha}.$$

Analogously we get

$$|u|_{C^\alpha(B_r(x))} \leq |\tilde{u}|_{C^\alpha(B_r(x))} \cdot K^\alpha.$$

We also have analogously for the first derivatives

$$|D\tilde{u}|_{C^0} \leq |Du|_{C^0} \cdot K_1$$

and

$$|Du|_{C^0} \leq |D\tilde{u}|_{C^0} \cdot K_1^{-1}$$

where K_1 depends on $|\Psi|_{C^1}$.

And for the second derivatives $\exists K_2$ depending on $|\Psi|_{C^2}$ (i.e depending on the domain Ω !) with

$$|D^2\tilde{u}| \leq K_1 \cdot |D^2u| + K_2|Du| \leq K'||u||_{C^2}.$$

Similarly

$$|D^2u| \leq K''||\tilde{u}||_{C^2}$$

Finally for $\mathcal{C}^{2,\alpha}$ norms, can again show norms on both sides are equivalent using K, K', K''. So we conclude - for each point $x \in \partial\Omega$ we have a ball and the estimate (1) holds *there*. The above long discussion yields that we have furthermore the desired estimate concerning the *norms* therein:

$$||u||_{C^{k,\beta}(B_{r/2}(x)\cap\Omega)} \leq c \cdot ||u||_{C^0(B_r(x)\cap\Omega)} + \epsilon \cdot |u|_{C^{2,\alpha}(B_{r/2}(x)\cap\Omega)}.$$

Since $\bar{\Omega}$ is compact we can cover $\partial\Omega$ by finitely many balls. On the interior of each we have this (desired) estimate. To complete the proof we therefore just need to make sure this estimate also holds in the interior of Ω. We take a set $\Omega' \Subset \Omega$ and a number $D > 0$ s.t.

A) if $x, y \in \Omega$ and $d(x, y) \leq D$ then either

    i) both x, y are both contained in one of the small ball covering the boundary of Ω. There we have the desired estimate already.

or

    ii) both x, y are in Ω'. Then we have the desired estimate as well from our interior estimate from the previous lecture for the semi-norms, which extends to give the desired estimate for the norms.

B) if $d(x, y) > D$ we have $\dfrac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq 1/D^\alpha \cdot 2 \cdot |D^2u|_{C^0(\Omega)}$. And so also in this case we get the desired estimate. ∎

We now try to extend our results to the case of non-constant coefficients , which was in fact our original goal.

Theorem. *Let $u \in \mathcal{C}^{2,\alpha}(\Omega)$, $L = a^{ij}(x)D_{ij} + b^i(x)D_i + c(x)$ (summation understood over double indices), and suppose $Lu = f \in \mathcal{C}^\alpha(\Omega)$. Assume furthermore that L has coefficients in $\mathcal{C}^\alpha(\Omega)$ and is uniformly elliptic, i.e*

- $\dfrac{1}{\Lambda} \cdot \delta^{ij} \leq a^{ij}(x) \leq \Lambda \cdot \delta^{ij}$

- $||a^{ij}(x)||_{C^\alpha(\Omega)},\ ||b^i(x)||_{C^\alpha(\Omega)},\ ||c(x)||_{C^\alpha(\Omega)} < \Lambda.$

Then $\forall\, \Omega' \Subset \Omega$, $\exists\, c = c(\Lambda, n, \Omega', \Omega)$ such that

$$||u||_{C^{2,\alpha}(\Omega')} \leq c\big(||u||_{C^0(\Omega)} + ||f||_{C^\alpha(\Omega)}\big).$$

Thanks to the interpolation proposition we really only need to bound the $\mathcal{C}^{2,\alpha}$ semi-norm with the above RHS since all the other semi-norms contained in the $\mathcal{C}^{2,\alpha}$ norm are bounded above by it together with the $\mathcal{C}^0$ semi-norm which is part of the RHS already.

We will try to make use of the Hölder constant of the coefficients to relate our situation to the constant coefficients case. The idea is that locally the coefficients are almost constant, and the degree to which this almost true is good enough for us (continuous wouldn't be good enough).

In this spirit, rewrite $Lu = f$ as

$$(a^{ij}(x) - a^{ij}(x_0) + a^{ij}(x_0))u_{ij}(x) + b^i(x)u_i(x) + c(x)u(x) = f,$$

or

$$a^{ij}(x_0)u_{ij}(x) = -(a^{ij}(x) - a^{ij}(x_0))u_{ij}(x) - b^i(x)u_i(x) - c(x)u(x) + f.$$

Calling the LHS $L_0 u$ and the RHS $F(x)$ we thus define a constant coefficient uniformly elliptic operator and a function.

Let $\Omega' \Subset \Omega$ with $x_0 \in \Omega'$. Take $B(x_0, \mu \cdot D) \Subset \Omega'$, $B(x_0, D) \Subset \Omega$ with $\mu > 0$ small and denote $d := \mu \cdot D$. As mentioned above we only need to prove an estimate for the $|D^2 u|_{C^\alpha(\Omega')}$. We observe that for any $y_0 \in \Omega'$, and for d small enough

$$\frac{D^2 u(x_0) - D^2 u(y_0)}{|x_0 - y_0|^\alpha} \leq \begin{cases} c \cdot (||u||_{C^0(B(x_0, \frac{d}{2}))} + ||F||_{C^\alpha(B(x_0, \frac{d}{2}))}), & \text{if } y_0 \in B(x_0, \frac{d}{2}) \\ \dfrac{2|u|_{C^2(\Omega')}}{d^\alpha} + ||F||_{C^0(\Omega')}, & \text{if } |x_0 - y_0| \geq \dfrac{d}{2}. \end{cases}$$

since when d is small enough, we can think of having a a uniform elliptic equation with almost constant coefficients in $B(x_0, \frac{d}{2})$ and then apply our previous results. We therefore get

$$|D^2 u|_{C^\alpha(\Omega')} \leq c' \cdot (||u||_{C^0(\Omega')} + \frac{1}{d^\alpha}|u|_{C^2(\Omega')} + \sup_{x_0 \in \Omega'} ||F||_{C^\alpha(B(x_0, \frac{d}{2}))}). \tag{2}$$

We first estimate the last term which is of a local nature. Observe that for two Hölder functions $k, l \in \mathcal{C}^\alpha(\Omega)$, $|k \cdot l|_{C^\alpha(\Omega)} \leq |k|_{C^\alpha(\Omega)} \cdot |l|_{C^0(\Omega)} + |k|_{C^0(\Omega)} \cdot |l|_{C^\alpha(\Omega)} \leq C(\Omega) \cdot |k|_{C^\alpha(\Omega)}|l|_{C^\alpha(\Omega)}$, while for the *norms* themselves we see from the first inequality that $||k \cdot l||_{C^\alpha(\Omega)} \leq ||k||_{C^\alpha(\Omega)} \cdot ||l||_{C^\alpha(\Omega)}$. So

$$||F||_{C^\alpha(B(x_0, \frac{d}{2}))} \leq ||a^{ij}(x) - a^{ij}(x_0)||_{C^\alpha(B(x_0, \frac{d}{2}))} \cdot ||u_{ij}||_{C^\alpha(B(x_0, \frac{d}{2}))} +$$

$$+ ||b^i||_{C^\alpha(B(x_0, \frac{d}{2}))} \cdot ||u_i||_{C^\alpha(B(x_0, \frac{d}{2}))} + ||c||_{C^\alpha(B(x_0, \frac{d}{2}))}||u||_{C^\alpha(B(x_0, \frac{d}{2}))} + ||f||_{C^\alpha(B(x_0, \frac{d}{2}))}.$$

4

Remember that the $\mathcal{C}^\alpha$ norm includes the $\mathcal{C}^0$ and $\mathcal{C}^\alpha$ semi-norms. We have

$$||a^{ij}(x)-a^{ij}(x_0)||_{C^\alpha(B(x_0,\frac{d}{2}))} = \sup|a^{ij}(x)-a^{ij}(x_0)|+|a^{ij}(x)-a^{ij}(x_0)|_{\mathcal{C}^\alpha} \le c\cdot||a^{ij}||_{C^\alpha(\Omega')}|x-x_0|^\alpha$$

$$\Longrightarrow$$

$$||F||_{C^\alpha(B(x_0,\frac{d}{2}))} \le c\cdot\Lambda d^\alpha(||\mathrm{D}^2u||_{C^\alpha(B(x_0,\frac{d}{2}))}+||\mathrm{D}u||_{C^\alpha(B(x_0,\frac{d}{2}))}+||u||_{C^\alpha(B(x_0,\frac{d}{2}))})+||f||_{C^\alpha(B(x_0,\frac{d}{2}))}$$

and using interpolation for the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ terms we find

$$\le c\cdot\Lambda d^\alpha(|u|_{C^{2,\alpha}(B(x_0,\frac{d}{2}))}+|u|_{C^0(\Omega')})+||f||_{C^\alpha(B(x_0,\frac{d}{2}))}. \tag{3}$$

We now come back to the 2$^{\text{nd}}$ term of (2).

We now let x_0 range over all points of Ω'. For each x_0 we will find different d, D such that (2) above holds.

It so happens that this inequality is useless unless we can not control from below the term involving $d^{-\alpha}$. The problem is that as $x_0 \to \partial\Omega$, $d = d(x_0) \to 0$. To overcome this we assume we chose Ω' such that $\Omega' \Subset \Omega'' \Subset \Omega$ and such that $\mathrm{dist}(\partial\Omega',\partial\Omega'') \ge \delta_0 > 0$. Then for problematic points $x_0 \to \partial\Omega$ and $y_0 \notin B(x_0,\frac{d}{2})$ we can replace the term $\frac{2|u|_{C^2(\Omega')}}{d^\alpha}$ by $\frac{2|u|_{C^2(\Omega')}}{(d+\delta_0)^\alpha}$. Therefore we have overcome the problem by means of introducing a constant c'' depending on Ω' (and Ω) and we can replace (2) by

$$|\mathrm{D}^2u|_{C^\alpha(\Omega')} \le c'\cdot(||u||_{C^0(\Omega')}+c''|u|_{C^2(\Omega'')}+\sup_{x_0\in\Omega'}||F||_{C^\alpha(B(x_0,\frac{d}{2}))}). \tag{4}$$

The 2$^{\text{nd}}$ term can be bounded above using our interpolation theory:

$$|u|_{C^2(\Omega'')} \le c(\epsilon_1)\cdot||u||_{C^0(\Omega)}+\epsilon_1\cdot|u|_{C^{2,\alpha}(\Omega'')} \tag{5}$$

with $\epsilon_1 > 0$ of our choice. And so we have all in all

$$|u|_{C^{2,\alpha}(\Omega')} \le c'\cdot(||u||_{C^0(\Omega)}+c(\epsilon_1)\cdot||u||_{C^0(\Omega)}+\epsilon_1\cdot|u|_{C^{2,\alpha}(\Omega'')}+c\Lambda d^\alpha(|u|_{C^{2,\alpha}(\Omega')}+|u|_{C^0(\Omega')})+||f||_{C^\alpha(\Omega')}).$$

Now choose d small enough such that $c\Lambda d^\alpha < \dfrac{1}{4}$ and $\epsilon_1 < \dfrac{1}{4}\cdot\dfrac{|u|_{C^{2,\alpha}(\Omega')}}{|u|_{C^{2,\alpha}(\Omega'')}}$. ∎

$\mathcal{R}emark.$ This Theorem gives still just a Schauder type *interior* estimate. Next time we will try to extend it to the boundary.