18. Solutions to (some of) the problems

Solution 18.1 (To Problem 10). (by Matjaž Konvalinka).

Since the topology on $\mathbb{N}$, inherited from $\mathbb{R}$, is discrete, a set is compact if and only if it is finite. If a sequence $\{x_n\}$ (i.e. a function $\mathbb{N} \to \mathbb{C}$) is in $\mathcal{C}_0(\mathbb{N})$ if and only if for any $\epsilon > 0$ there exists a compact (hence finite) set F_ϵ so that $|x_n| < \epsilon$ for any n not in F_ϵ. We can assume that $F_\epsilon = \{1, \dots, n_\epsilon\}$, which gives us the condition that $\{x_n\}$ is in $\mathcal{C}_0(\mathbb{N})$ if and only if it converges to 0. We denote this space by c_0, and the supremum norm by $\|\cdot\|_0$. A sequence $\{x_n\}$ will be abbreviated to x.

Let l^1 denote the space of (real or complex) sequences x with a finite 1-norm

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|.$$

We can define pointwise summation and multiplication with scalars, and $(l^1, \|\cdot\|_1)$ is a normed (in fact Banach) space. Because the functional

$$y \mapsto \sum_{n=1}^{\infty} x_n y_n$$

is linear and bounded ($|\sum_{n=1}^{\infty} x_n y_n| \le \sum_{n=1}^{\infty} |x_n||y_n| \le \|x\|_0 \, \|y\|_1$) by $\|x\|_0$, the mapping

$$\Phi \colon l^1 \longmapsto c_0^*$$

defined by

$$x \mapsto \left(y \mapsto \sum_{n=1}^{\infty} x_n y_n \right)$$

is a (linear) well-defined mapping with norm at most 1. In fact, Φ is an isometry because if $|x_j| = \|x\|_0$ then $|\Phi(x)(e_j)| = 1$ where e_j is the j-th unit vector. We claim that Φ is also surjective (and hence an isometric isomorphism). If φ is a functional on c_0 let us denote $\varphi(e_j)$ by x_j. Then $\Phi(x)(y) = \sum_{n=1}^{\infty} \varphi(e_n) y_n = \sum_{n=1}^{\infty} \varphi(y_n e_n) = \varphi(y)$ (the last equality holds because $\sum_{n=1}^{\infty} y_n e_n$ converges to y in c_0 and φ is continuous with respect to the topology in c_0), so $\Phi(x) = \varphi$.

Solution 18.2 (To Problem 29). (Matjaž Konvalinka) Since

$$D_x H(\varphi) = H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x)\, dx = \\ i \int_0^{\infty} \varphi'(x)\, dx = i(0 - \varphi(0)) = -i\delta(\varphi),$$

we get $D_x H = C\delta$ for $C = -i$.

Solution 18.3 (To Problem 40). (Matjaž Konvalinka) Let us prove this in the case where $n = 1$. Define (for $b \neq 0$)

$$U(x) = u(b) - u(x) - (b-x)u'(x) - \ldots - \frac{(b-x)^{k-1}}{(k-1)!}u^{(k-1)}(x);$$

then

$$U'(x) = -\frac{(b-x)^{k-1}}{(k-1)!}u^{(k)}(x).$$

For the continuously differentiable function $V(x) = U(x) - (1-x/b)^k U(0)$ we have $V(0) = V(b) = 0$, so by Rolle's theorem there exists ζ between 0 and b with

$$V'(\zeta) = U'(\zeta) + \frac{k(b-\zeta)^{k-1}}{b^k}U(0) = 0$$

Then

$$U(0) = -\frac{b^k}{k(b-\zeta)^{k-1}}U'(\zeta),$$

$$u(b) = u(0) + u'(0)b + \ldots + \frac{u^{(k-1)}(0)}{(k-1)!}b^{k-1} + \frac{u^{(k)}(\zeta)}{k!}b^k.$$

The required decomposition is $u(x) = p(x) + v(x)$ for

$$p(x) = u(0) + u'(0)x + \frac{u''(0)}{2}x^2 + \ldots + \frac{u^{(k-1)}(0)}{(k-1)!}x^{k-1} + \frac{u^{(k)}(0)}{k!}x^k,$$

$$v(x) = u(x) - p(x) = \frac{u^{(k)}(\zeta) - u^{(k)}(0)}{k!}x^k$$

for ζ between 0 and x, and since $u^{(k)}$ is continuous, $(u(x) - p(x))/x^k$ tends to 0 as x tends to 0.

The proof for general n is not much more difficult. Define the function $w_x \colon I \to \mathbb{R}$ by $w_x(t) = u(tx)$. Then w_x is k-times continuously differentiable,

$$w'_x(t) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(tx)x_i,$$

$$w''_x(t) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(tx)x_i x_j,$$

$$w_x^{(l)}(t) = \sum_{l_1+l_2+\ldots+l_i=l} \frac{l!}{l_1!l_2!\cdots l_i!}\frac{\partial^l u}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_i^{l_i}}(tx)x_1^{l_1}x_2^{l_2}\cdots x_i^{l_i}$$

so by above $u(x) = w_x(1)$ is the sum of some polynomial p (od degree k), and we have

$$\frac{u(x)-p(x)}{|x|^k} = \frac{v_x(1)}{|x|^k} = \frac{w_x^{(k)}(\zeta_x) - w_x^{(k)}(0)}{k!|x|^k},$$

so it is bounded by a positive combination of terms of the form

$$\left|\frac{\partial^l u}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_i^{l_i}}(\zeta_x x) - \frac{\partial^l u}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_i^{l_i}}(0)\right|$$

with $l_1 + \ldots + l_i = k$ and $0 < \zeta_x < 1$. This tends to zero as $x \to 0$ because the derivative is continuous.

Solution 18.4 (Solution to Problem 41). (Matjž Konvalinka) Obviously the map $\mathcal{C}_0(\mathbb{B}^n) \to \mathcal{C}(\mathbb{B}^n)$ is injective (since it is just the inclusion map), and $f \in \mathcal{C}(\mathbb{B}^n)$ is in $\mathcal{C}_0(\mathbb{B}^n)$ if and only if it is zero on $\partial\mathbb{B}^n$, ie. if and only if $f|_{\mathbb{S}^{n-1}} = 0$. It remains to prove that any map g on $\mathbb{S}^{n-1}$ is the restriction of a continuous function on $\mathbb{B}^n$. This is clear since

$$f(x) = \begin{cases} |x|g(x/|x|) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is well-defined, coincides with f on $\mathbb{S}^{n-1}$, and is continuous: if M is the maximum of $|g|$ on $\mathbb{S}^{n-1}$, and $\epsilon > 0$ is given, then $|f(x)| < \epsilon$ for $|x| < \epsilon/M$.

Solution 18.5. (partly Matjaž Konvalinka)
For any $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}|\int_{-\infty}^{\infty}\varphi(x)dx| \le \int_{-\infty}^{\infty}|\varphi(x)|dx \le \sup((1+x|^2)|\varphi(x)|)\int_{-\infty}^{\infty}(1+|x|^2)^{-1}dx \\ \le C\sup((1+x|^2)|\varphi(x)|).\end{aligned}$$

Thus $\mathcal{S}(\mathbb{R}) \ni \varphi \longmapsto \int_{\mathbb{R}}\varphi dx$ is continous.

Now, choose $\phi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R})$ with $\int_{\mathbb{R}}\phi(x)dx = 1$. Then, for $\psi \in \mathcal{S}(\mathbb{R})$, set

$$A\psi(x) = \int_{-\infty}^{x}(\psi(t) - c(\psi)\phi(t))\,dt,\ c(\psi) = \int_{-\infty}^{\infty}\psi(s)\,ds. \tag{18.1}$$

Note that the assumption on ϕ means that

$$A\psi(x) = -\int_{x}^{\infty}(\psi(t) - c(\psi)\phi(t))\,dt \tag{18.2}$$

Clearly $A\psi$ is smooth, and in fact it is a Schwartz function since

$$\frac{d}{dx}(A\psi(x)) = \psi(x) - c\phi(x) \in \mathcal{S}(\mathbb{R}) \tag{18.3}$$

so it suffices to show that $x^kA\psi$ is bounded for any k as $|x| \to \pm\infty$. Since $\psi(t) - c\phi(t) \le C_k t^{-k-1}$ in $t \ge 1$ it follows from (18.2) that

$$|x^kA\psi(x)| \le Cx^k \int_x^\infty t^{-k-1}dt \le C',\ k > 1,\ \text{in } x > 1.$$

A similar estimate as $x \to -\infty$ follows from (18.1). Now, A is clearly linear, and it follows from the estimates above, including that on the integral, that for any k there exists C and j such that

$$\sup_{\alpha,\beta\le k} |x^\alpha D^\beta A\psi| \le C \sum_{\alpha',\beta'\le j} \sup_{x\in\mathbb{R}} |x^{\alpha'} D^{\beta'}\psi|.$$

Finally then, given $u \in \mathcal{S}'(\mathbb{R})$ define $v(\psi) = -u(A\psi)$. From the continuity of A, $v \in \mathcal{S}(\mathbb{R})$ and from the definition of A, $A(\psi') = \psi$. Thus

$$dv/dx(\psi) = v(-\psi') = u(A\psi') = u(\psi) \Longrightarrow \frac{dv}{dx} = u.$$

Solution 18.6. We have to prove that $\langle\xi\rangle^{m+m'}\widehat{u} \in L_2(\mathbb{R}^n)$, in other words, that

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2(m+m')}|\widehat{u}|^2\,d\xi < \infty.$$

But that is true since

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2(m+m')}|\widehat{u}|^2\,d\xi = \int_{\mathbb{R}^n} \langle\xi\rangle^{2m'}(1+\xi_1^2+\ldots+\xi_n^2)^m|\widehat{u}|^2\,d\xi =$$

$$= \int_{\mathbb{R}^n} \langle\xi\rangle^{2m'}\left(\sum_{|\alpha|\le m} C_\alpha\xi^{2\alpha}\right)|\widehat{u}|^2\,d\xi = \sum_{|\alpha|\le m} C_\alpha \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2m'}\xi^{2\alpha}|\widehat{u}|^2\,d\xi\right)$$

and since $\langle\xi\rangle^{m'}\xi^\alpha\widehat{u} = \langle\xi\rangle^{m'}\widehat{D^\alpha u}$ is in $L^2(\mathbb{R}^n)$ (note that $u \in H^m(\mathbb{R}^n)$ follows from $D^\alpha u \in H^{m'}(\mathbb{R}^n)$, $|\alpha| \le m$). The converse is also true since C_α in the formula above are strictly positive.

Solution 18.7. Take $v \in L^2(\mathbb{R}^n)$, and define subsets of $\mathbb{R}^n$ by

$$E_0 = \{x\colon |x| \le 1\},$$
$$E_i = \{x\colon |x| \ge 1, |x_i| = \max_j |x_j|\}.$$

Then obviously we have $1 = \sum_{i=0}^n \chi_{E_j}$ a.e., and $v = \sum_{j=0}^n v_j$ for $v_j = \chi_{E_j}v$. Then $\langle x\rangle$ is bounded by $\sqrt{2}$ on E_0, and $\langle x\rangle v_0 \in L^2(\mathbb{R}^n)$; and on E_j, $1 \le j \le n$, we have

$$\frac{\langle x\rangle}{|x_j|} \le \frac{(1+n|x_j|^2)^{1/2}}{|x_j|} = \left(n + 1/|x_j|^2\right)^{1/2} \le (2n)^{1/2},$$

so $\langle x\rangle v_j = x_j w_j$ for $w_j \in L^2(\mathbb{R}^n)$. But that means that $\langle x\rangle v = w_0 + \sum_{j=1}^n x_j w_j$ for $w_j \in L^2(\mathbb{R}^n)$.
If u is in $L^2(\mathbb{R}^n)$ then $\widehat{u} \in L^2(\mathbb{R}^n)$, and so there exist $w_0, \dots, w_n \in L^2(\mathbb{R}^n)$ so that

$$\langle\xi\rangle\widehat{u} = w_0 + \sum_{j=1}^n \xi_j w_j,$$

in other words

$$\widehat{u} = \widehat{u}_0 + \sum_{j=1}^n \xi_j \widehat{u}_j$$

where $\langle\xi\rangle\widehat{u}_j \in L^2(\mathbb{R}^n)$. Hence

$$u = u_0 + \sum_{j=1}^n D_j u_j$$

where $u_j \in H^1(\mathbb{R}^n)$.

Solution 18.8. Since

$$D_x H(\varphi) = H(-D_x\varphi) = i\int_{-\infty}^{\infty} H(x)\varphi'(x)\,dx = i\int_0^{\infty}\varphi'(x)\,dx = i(0-\varphi(0)) = -i\delta(\varphi),$$

we get $D_x H = C\delta$ for $C = -i$.

Solution 18.9. It is equivalent to ask when $\langle\xi\rangle^m\widehat{\delta_0}$ is in $L^2(\mathbb{R}^n)$. Since

$$\widehat{\delta_0}(\psi) = \delta_0(\widehat{\psi}) = \widehat{\psi}(0) = \int_{\mathbb{R}^n}\psi(x)\,dx = 1(\psi),$$

this is equivalent to finding m such that $\langle\xi\rangle^{2m}$ has a finite integral over $\mathbb{R}^n$. One option is to write $\langle\xi\rangle = (1+r^2)^{1/2}$ in spherical coordinates, and to recall that the Jacobian of spherical coordinates in n dimensions has the form $r^{n-1}\Psi(\varphi_1, \dots, \varphi_{n-1})$, and so $\langle\xi\rangle^{2m}$ is integrable if and only if

$$\int_0^{\infty}\frac{r^{n-1}}{(1+r^2)^m}\,dr$$

converges. It is obvious that this is true if and only if $n-1-2m < -1$, ie. if and only if $m > n/2$.

Solution 18.10 (Solution to Problem31). We know that $\delta \in H^m(\mathbb{R}^n)$ for any $m < -n/1$. Thus is just because $\langle\xi\rangle^p \in L^2(\mathbb{R}^n)$ when $p < -n/2$. Now, divide $\mathbb{R}^n$ into $n+1$ regions, as above, being $A_0 = \{\xi; |\xi| \le 1$ and $A_i = \{\xi; |\xi_i| = \sup_j |\xi_j|,\ |\xi| \ge 1\}$. Let v_0 have Fourier transform χ_{A_0} and for $i = 1, \dots, n$, $v_i \in \mathcal{S};(\mathbb{R}^n)$ have Fourier transforms $\xi_i^{-n-1}\chi_{A_i}$. Since $|\xi_i| > c\langle\xi\rangle$ on the support of $\widehat{v}_i$ for each $i = 1, \dots, n$, each term is in H^m for any $m < 1 + n/2$ so, by the Sobolev embedding theorem, each $v_i \in \mathcal{C}_0^0(\mathbb{R}^n)$ and

$$1 = \hat{v}_0 \sum_{i=1}^{n} \xi_i^{n+1} \widehat{v_i} \Longrightarrow \delta = v_0 + \sum_i D_i^{n+1} v_i. \tag{18.4}$$

How to see that this cannot be done with n or less derivatives? For the moment I do not have a proof of this, although I believe it is true. Notice that we are actually proving that δ can be written

$$\delta = \sum_{|\alpha| \le n+1} D^\alpha u_\alpha,\ u_\alpha \in H^{n/2}(\mathbb{R}^n). \tag{18.5}$$

This cannot be improved to n from $n+1$ since this would mean that $\delta \in H^{-n/2}(\mathbb{R}^n)$, which it isn't. However, what I am asking is a little more subtle than this.