18.152 Introduction to PDEs, Fall 2011   Professor: Jared Speck

Class Meeting # 21: Lagrangian Field Theories

Many of the PDEs of interest to us can be realized as the *Euler-Lagrange* equations corresponding to a function known as a *Lagrangian* $\mathcal{L}$. Closely related are the notions of the *action* corresponding to the Lagrangian, and the notion of a *stationary point* of the action. These ideas fall under a branch of mathematics known as the calculus of variations. As we will see, these ideas will provide a framework for deriving conserved (and more generally almost-conserved) quantities for solutions to the Euler-Lagrange equations, the availability of which plays a central role in the analysis of these solutions. Some important examples of PDEs to which these methods apply include the familiar linear wave equation, Maxwell's equations of electromagnetism, the Euler equations of fluid mechanics, and the Einstein equations of general relativity.

1. Variational formulation (The Action Principle)

In this section, we will study (scalar-valued) functions ϕ on $\mathbb{R}^{1+n}$. They are sometimes called (scalar-valued) *fields* on $\mathbb{R}^{1+n}$. We will use the notation

$$(1.0.1) \qquad x = (x^0, x^1, \cdots, x^n)$$

to denote the standard coordinates on $\mathbb{R}^{1+n}$, and as usual, we will sometimes use the alternate notation $x^0 = t$. We will use the notation

$$(1.0.2) \qquad \nabla \phi \overset{\text{def}}{=} (\nabla_t \phi, \nabla_1 \phi, \cdots, \nabla_n \phi)$$

to denote the spacetime gradient of ϕ. We will study PDEs that are (in a sense to be explained) generated by a Lagrangian.

Definition 1.0.1 (Lagrangian). A Lagrangian $\mathcal{L}$ is a function of ϕ and $\nabla \phi$ (and sometimes the spacetime coordinates x and perhaps other quantities too). We indicate the dependence of $\mathcal{L}$ on e.g. ϕ and $\nabla \phi$ by writing

$$(1.0.3) \qquad \mathcal{L}(\phi, \nabla \phi).$$

Example 1.0.1. As we will see, $\mathcal{L} \overset{\text{def}}{=} \frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$, is the Lagrangian corresponding to the linear wave equation, where $m^{-1} = \text{diag}(-1, 1, 1, \cdots, 1)$ is the standard Minkowski metric.

Given a Lagrangian $\mathcal{L}$ and a compact subset of spacetime $\mathfrak{K}$, we can define an important *functional* known as the *action*. The action inputs functions ϕ and outputs a real number.

Definition 1.0.2 (Action). Let $\mathfrak{K} \subset \mathbb{R}^{1+n}$ be a compact subset of spacetime. We define the action $\mathcal{A}$ of ϕ over the set $\mathfrak{K}$ by

$$(1.0.4) \qquad \mathcal{A}[\phi; \mathfrak{K}] \overset{\text{def}}{=} \int_{\mathfrak{K}} \mathcal{L}(\phi(x), \nabla\phi(x)) \, d^{1+n}x.$$

Above, $d^{1+n}x \overset{\text{def}}{=} dt dx^1 d^2 \cdots dx^n$ denotes spacetime integration. We often omit the argument x of ϕ and $\nabla\phi$.

A main theme that runs throughout this section is that it is possible to generalize certain aspects of standard calculus, which takes place on $\mathbb{R}^{1+n}$, to apply to (infinite dimensional) spaces of functions. In this context, the action $\mathcal{A}$ plays the same role that a function plays in standard calculus. Moreover, many important PDEs have solutions that are *stationary points*[1] for the action. The notion of a stationary point is a generalization of the notion of a critical point from calculus. In order to define a stationary point of $\mathcal{A}$, we will need to introduce the notion of a *variation*. The motivation behind the next two definitions is that we would like to understand how $\mathcal{A}[\phi; \mathfrak{K}]$ changes when we slightly change ϕ.

Definition 1.0.3 (Variation). Given a compact set $\mathfrak{K}$, a function $\psi \in C_c^\infty(\mathfrak{K})$ is called a variation.

Definition 1.0.4. Given a variation ψ and a small number ϵ, we define

$$(1.0.5) \qquad \phi_\epsilon \overset{\text{def}}{=} \phi + \underbrace{\epsilon\psi}_{\text{tiny perturbation of } \phi}.$$

We now give the definition of a stationary point of the action. Stationary points are the moral equivalent of critical points[2] from calculus.

Definition 1.0.5 (Definition of a stationary point ϕ). We say that ϕ is a **stationary point** of the action if the following relation holds for all compact subsets $\mathfrak{K}$ and all variations $\psi \in C_c^\infty(\mathfrak{K})$:

$$(1.0.6) \qquad \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{A}[\phi_\epsilon; \mathfrak{K}] = 0.$$

The next theorem is central to our discussion in this section. It shows that the stationary points of $\mathcal{A}$ verify a PDE called the *Euler-Lagrange equation*.

Theorem 1.1 (The Principle of Stationary Action). *Let $\mathcal{L}(\phi, \nabla\phi, x)$ be a C^2 Lagrangian. Then a C^2 field ϕ is a stationary point of the action if and only if the following **Euler-Lagrange** PDE is verified by ϕ :*

$$(1.0.7) \qquad \nabla_\alpha \left(\frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial(\nabla_\alpha \phi)} \right) = \frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial \phi}.$$

Above, $\frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial(\nabla_\alpha \phi)}$ denotes partial differentiation of $\mathcal{L}$ with respect to its argument $\nabla_\alpha \phi$ with its other arguments (e.g., the other $\nabla_\mu \phi$ with $\mu \neq \alpha$, ϕ, x, etc.) held fixed.

Proof. Let $\mathfrak{K} \subset \mathbb{R}^{1+n}$ be a compact subset of spacetime and let ψ be any variation with support contained in $\mathfrak{K}$. For any $\epsilon > 0$, we define as in (1.0.5): $\phi_\epsilon \overset{\text{def}}{=} \phi + \epsilon\psi$. We then differentiate under the integral and use the chain rule to conclude that

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[1]Even though they are called "stationary points," they are actually fields on $\mathbb{R}^{1+n}$.

[2]Recall that x is a critical point of the function f if $f'(x) = 0$.

$$(1.0.8) \qquad \frac{d}{d\epsilon}\mathcal{A}[\phi_\epsilon; \mathfrak{K}] \overset{\text{def}}{=} \frac{d}{d\epsilon}\int_{\mathfrak{K}} \mathcal{L}(\phi_\epsilon, \nabla\phi_\epsilon, x)\, d^{1+n}x = \int_{\mathfrak{K}} \partial_\epsilon \mathcal{L}(\phi_\epsilon, \nabla\phi_\epsilon, x)\, d^{1+n}x$$

$$= \int_{\mathfrak{K}} \frac{\partial \mathcal{L}(\phi_\epsilon, \nabla\phi_\epsilon, x)}{\partial \phi} \underbrace{\partial_\epsilon \phi_\epsilon}_{\psi} + \frac{\partial \mathcal{L}(\phi_\epsilon, \nabla\phi_\epsilon, x)}{\partial(\nabla_\alpha \phi)} \underbrace{\partial_\epsilon \nabla_\alpha \phi_\epsilon}_{\nabla_\alpha \psi}\, d^{1+n}x.$$

Above, ∂_ϵ denotes the derivative with respect to the parameter ϵ with all other variables held fixed. We now set $\epsilon = 0$, integrate by parts in (1.0.8) (and observe that the conditions on ψ guarantee that there are no boundary terms) to deduce that

$$(1.0.9) \qquad \frac{d}{d\epsilon}\mathcal{A}[\phi_\epsilon; \mathfrak{K}] = \int_{\mathfrak{K}} \frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial \phi}\psi + \frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial(\nabla_\alpha \phi)}\nabla_\alpha \psi\, d^{1+n}x$$

$$= \int_{\mathfrak{K}} \frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial \phi}\psi - \nabla_\alpha \left(\frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial(\nabla_\alpha \phi)} \right) \psi\, d^{1+n}x$$

$$= \int_{\mathfrak{K}} \left\{ \frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial \phi} - \nabla_\alpha \left(\frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial(\nabla_\alpha \phi)} \right) \right\} \psi\, d^{1+n}x.$$

We now observe that (1.0.9) is equal to 0 for all variations ψ if and only if the term in large brackets on the right-hand side of (1.0.9) must be 0. Since this observation holds for any compact subset $\mathfrak{K}$, we have thus shown that (1.0.7) holds if and only if ϕ is a stationary point of the action.

$\square$

Example 1.0.2. Let $\mathcal{L} \overset{\text{def}}{=} -\frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$ (note that this $\mathcal{L}$ does not directly depend on x) where $m^{-1} = \text{diag}(-1, 1, 1, \cdots, 1)$ is the standard Minkowski metric. Then

$$(1.0.10) \qquad \frac{\partial \mathcal{L}(\phi, \nabla\phi)}{\partial \phi} = 0,$$

$$(1.0.11) \qquad \frac{\partial \mathcal{L}(\phi, \nabla\phi)}{\partial(\nabla_\mu \phi)} = -(m^{-1})^{\mu\alpha}\nabla_\alpha\phi.$$

Therefore, the Euler-Lagrange equation corresponding to $\mathcal{L}$ is

$$(1.0.12) \qquad \nabla_\mu\left((m^{-1})^{\mu\alpha}\nabla_\alpha\phi \right) = 0.$$

Note that equation (1.0.12) is just the familiar linear wave equation $(m^{-1})^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0$.

2. COORDINATE INVARIANT LAGRANGIANS

Many important PDEs are the Euler-Lagrange equations corresponding to *coordinate invariant* Lagrangians; we will explain what this means momentarily. Motivated by this claim, we will now introduce a class of changes of coordinates on spacetime. The new coordinates will be formed by flowing the old coordinates in the direction of a vectorfield Y on spacetime. These new coordinates will therefore verify a system of ordinary differential equations generated by the flow of Y. In the next proposition, we review some facts concerning these new coordinates; these facts are basic results in ODE theory.

Proposition 2.0.1 (Basic facts from ODE theory for autonomous systems). *Let* $Y(x) = \big(Y^0(\underbrace{x^0, \cdots, x^n}_{x}), Y^1(x^0, \cdots, x^n), \cdots, Y^n(x^0, \cdots, x^n)\big)$ *be a smooth vectorfield on* $\mathbb{R}^{1+n}$. *Assume that there exists a uniform constant* $C > 0$ *such that*

$$(2.0.13) \qquad\qquad |\nabla_\mu Y^\nu(x)| \leq C, \qquad x \in \mathbb{R}^{1+n}, \qquad 0 \leq \mu, \nu \leq n.$$

Consider the initial value problem (where the independent variable is the "flow parameter" ϵ) for the following system of ordinary differential equations:

$$(2.0.14) \qquad\qquad \frac{d}{d\epsilon}\widetilde{x}^\mu(\epsilon) = Y^\mu(\widetilde{x}),$$

$$(2.0.15) \qquad\qquad \widetilde{x}^\mu(0) = x^\mu.$$

Then there exists a number $\epsilon_0 > 0$ such that the initial value problem (2.0.14) - (2.0.15) has a unique smooth (in ϵ) solution existing on the interval $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Let us denote the "flow map" from the data x to the solution $\widetilde{x}$ at flow parameter ϵ by $\widetilde{x} = F_\epsilon(x)$. Then on the interval $[-\epsilon_0, \epsilon_0]$, the flow map

$$(2.0.16) \qquad\qquad x \to F_\epsilon(x) \overset{def}{=} \widetilde{x}.$$

*is a smooth (in x), bijective map from $\mathbb{R}^{1+n}$ to $\mathbb{R}^{1+n}$ with smooth inverse $F_{-\epsilon}(\cdot)$, i.e., $\widetilde{x} = F_\epsilon(x) \implies x = F_{-\epsilon}(\widetilde{x})$; such maps are called **diffeomorphisms** of $\mathbb{R}^{1+n}$. Furthermore, if $|\epsilon_1| + |\epsilon_2| \leq \epsilon_0$, then the flow map verifies the following **one-parameter commutative group** properties:*

$$(2.0.17) \qquad\qquad F_{\epsilon_1} \circ F_{\epsilon_2} = F_{\epsilon_2} \circ F_{\epsilon_1} = F_{\epsilon_1+\epsilon_2}.$$

Let us also denote the derivative matrix corresponding to the flow map F_ϵ by M :

$$(2.0.18) \qquad\qquad M^\mu_\nu \overset{def}{=} \frac{\partial \widetilde{x}^\mu}{\partial x^\nu}.$$

Then if $|\epsilon|$ is sufficiently small, we have the following expansions in ϵ :

$$(2.0.19) \qquad\qquad \widetilde{x}^\mu \overset{def}{=} F^\mu_\epsilon(x) = x^\mu + \epsilon Y^\mu(x) + \epsilon^2 \mathcal{R}^\mu(\epsilon, x),$$

$$(2.0.20) \qquad\qquad M^\mu_\nu = \delta^\mu_\nu + \epsilon \nabla_\nu Y^\mu(x) + \epsilon^2 \nabla_\nu \mathcal{R}^\mu(\epsilon, x),$$

$$(2.0.21) \qquad\qquad (M^{-1})^\mu_\nu = \frac{\partial x^\mu}{\partial \widetilde{x}^\nu} = \delta^\mu_\nu - \epsilon \nabla_\nu Y^\mu(x) + \epsilon^2 \mathcal{S}^\mu_\nu(\epsilon, x),$$

$$(2.0.22) \qquad\qquad det M^{-1} = 1 - \epsilon \nabla_\alpha Y^\alpha + \epsilon^2 \mathcal{S}(\epsilon, x).$$

Above, $\mathcal{R}^\mu(\epsilon, x), \nabla_\nu \mathcal{R}^\mu(\epsilon, x), \mathcal{S}^\mu_\nu(\epsilon, x), \mathcal{S}(\epsilon, x)$ are smooth functions of (ϵ, x) for $\epsilon \in [-\epsilon_0, \epsilon_0]$, $x \in \mathbb{R}^{1+n}$.

Remark 2.0.1. The assumption (2.0.13) guarantees that the "time of existence" ϵ_0 can be chosen to be independent of the initial data x.

Proof. Most of the results of Proposition 2.0.1 are standard facts from ODE theory and will not be proved here. We will show how to derive the expansions (2.0.21) and (2.0.22) from the other results. To this end, we will need some basic facts from matrix theory. We will use the following norm for $(1+n) \times (1+n)$ matrix-valued functions on $\mathbb{R}^{1+n}$:

$$(2.0.23) \qquad \|M\| \overset{\text{def}}{=} \max_{x \in \mathbb{R}^{1+n}} \sqrt{\sum_{0 \le \mu, \nu \le n} |M_\nu^\mu(x)|^2}.$$

Now if I is the $1+n$ identity matrix[3], and $\|A\|$ is a sufficiently small $(1+n) \times (1+n)$ matrix, then the matrix $M \overset{\text{def}}{=} (I-A)^{-1}$ can be expanded in a convergent series:

$$(2.0.24) \qquad (I-A)^{-1} = I + A + A^2 + A^3 + \cdots .$$

Note in particular that the tail (i.e., all but the first two terms) can be bounded by

$$(2.0.25) \qquad \|A^2 + A^3 + A^4 + \cdots \| = \|A^2(I-A)^{-1}\| \le 2\|A\|^2,$$

if $\|A\|$ is sufficiently small. We now apply (2.0.24) and (2.0.25) to the matrix M defined in (2.0.18) (where $A_\nu^\mu \overset{\text{def}}{=} \epsilon \nabla_\nu Y^\mu$), thereby arriving at (2.0.21).

To derive (2.0.22), we first Taylor expand the determinant (viewed as a real-valued function of matrices) for sufficiently small $\|A\|$:

$$(2.0.26) \qquad \det(I+A) = 1 + A_\alpha^\alpha + O(\|A\|^2)$$

Above, we write $O(\|A\|^2)$ to denote a term that can be bounded by $C\|A\|^2$, where $C > 0$ is some positive constant independent of (all sufficiently small) A. The expansion (2.0.22) now follows from (2.0.21) and (2.0.26). We remark that you will derive the expansion (2.0.26) in your homework in more detail.

$\square$

We will now "define" how various fields and their derivatives transform under a change of coordinates. A full justification of these definitions can be found in books on tensor analysis or differential geometry.

Definition 2.0.6 (Transformation properties of fields). Let $\phi(x)$ be a scalar-valued function, let $m(x)$ be an (invertible) metric (depending on x) with components $m_{\mu\nu}(x)$, and let $x \to \widetilde{x}$ be a spacetime diffeomorphism. Then upon changing coordinates $x \to \widetilde{x}$, these quantities transform as follows:

$$(2.0.27a) \qquad \widetilde{\phi}(\widetilde{x}) \overset{\text{def}}{=} \phi|_{(x \circ \widetilde{x})},$$

$$(2.0.27b) \qquad \widetilde{\nabla}_\mu \widetilde{\phi}(\widetilde{x}) \overset{\text{def}}{=} (M^{-1})_\mu^\alpha|_{(x \circ \widetilde{x})} \nabla_\alpha \phi|_{x \circ \widetilde{x}},$$

$$(2.0.27c) \qquad \widetilde{m}_{\mu\nu}(\widetilde{x}) \overset{\text{def}}{=} (M^{-1})_\mu^\alpha|_{(x \circ \widetilde{x})} (M^{-1})_\nu^\beta|_{(x \circ \widetilde{x})} m_{\alpha\beta}|_{(x \circ \widetilde{x})},$$

$$(2.0.27d) \qquad (\widetilde{m}^{-1})^{\mu\nu}(\widetilde{x}) \overset{\text{def}}{=} M_\alpha^\mu|_{(x \circ \widetilde{x})} M_\beta^\nu|_{(x \circ \widetilde{x})} (m^{-1})^{\alpha\beta}|_{(x \circ \widetilde{x})}.$$

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[3]Note that $I_\nu^\mu = \delta_\nu^\mu$.

Above and throughout, we use the notation

(2.0.28a)
$$\nabla_\mu \overset{\text{def}}{=} \frac{\partial}{\partial x^\mu},$$

(2.0.28b)
$$\widetilde{\nabla}_\mu \overset{\text{def}}{=} \frac{\partial}{\partial \widetilde{x}^\mu},$$

$M^\mu_\nu \overset{\text{def}}{=} \frac{\partial \widetilde{x}^\mu}{\partial x^\nu}$ is the derivative matrix defined in (2.0.18), and $(M^{-1})^\mu_\nu = \frac{\partial x^\mu}{\partial \widetilde{x}^\nu}$ is its inverse. Furthermore, the notation $x \circ \widetilde{x}$ indicates that we are viewing x as a function of $\widetilde{x}$; this is possible since $x \to \widetilde{x}$ is a diffeomorphism.

Remark 2.0.2. (2.0.27a) simply says that the transformed function $\widetilde{\phi}$ takes the same value at the new coordinate $\widetilde{x}$ that ϕ takes at the old coordinate x. (2.0.27b) is really just the chain rule expressing $\frac{\partial}{\partial \widetilde{x}^\mu}$ in terms of $\frac{\partial}{\partial x^\mu}$. (2.0.27c) is the standard transformation law for *tensors* with two upstairs indices. These transformation laws generalize to other tensors in a straightforward fashion; the generalization can be found in books on tensor analysis/ differential geometry. Roughly speaking, tensors with indices downstairs transform by multiplication by the matrix M^{-1} (one copy of M^{-1} for each index), and tensors with indices upstairs transform by multiplication by the matrix M (one copy of M for each index).

We will now define what it means for a Lagrangian to be coordinate invariant.

Definition 2.0.7 (**Coordinate invariant Lagrangian**). Let $\mathcal{L}(\phi, \nabla\phi, m)$ be a Lagrangian that depends *only* on ϕ, $\nabla\phi$, and the Minkowski metric m. We say that $\mathcal{L}$ is *coordinate invariant* if for all spacetime diffeomorphisms $x \to \widetilde{x}$, we have that

(2.0.29)
$$\mathcal{L}(\phi(x), \nabla\phi(x), m(x)) = \mathcal{L}(\widetilde{\phi}(\widetilde{x}), \widetilde{\nabla}\widetilde{\phi}(\widetilde{x}), \widetilde{m}(\widetilde{x})),$$

where the transformed fields are defined in Definition 2.0.6.

Example 2.0.3. Consider the Lagrangian for the linear wave equation: $\mathcal{L}(\phi, \nabla\phi, m) = -\frac{1}{2}(m^{-1})^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. Using (2.0.27a) - (2.0.27c) and the fact that $M^\mu_\alpha (M^{-1})^\kappa_\mu = \delta^\kappa_\alpha$[4], we compute that

(2.0.30)
$$\begin{aligned}
\mathcal{L}(\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m}) &\overset{\text{def}}{=} -\frac{1}{2}(\widetilde{m}^{-1})^{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\phi}\widetilde{\nabla}_\nu\widetilde{\phi} \\
&= -\frac{1}{2}M^\mu_\alpha M^\nu_\beta (m^{-1})^{\alpha\beta}(M^{-1})^\kappa_\mu \nabla_\kappa\phi (M^{-1})^\lambda_\mu \nabla_\lambda\phi \\
&= -\frac{1}{2}(m^{-1})^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi.
\end{aligned}$$

This Lagrangian is therefore coordinate invariant.

As we will see, the availability of an energy-momentum tensor for certain Euler-Lagrange equations is closely connected to the coordinate invariance property of their Lagrangians. In order to derive this connection, we will need to understand more about how the coordinate transformations (2.0.16) vary with ϵ.

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[4]Recall that $\delta^\kappa_\alpha = 1$ if $\alpha = \kappa$ and $\delta^\kappa_\alpha = 1$ if $\alpha \neq \kappa$; δ^κ_α can be viewed as the identity matrix.

Proposition 2.0.2 (**Derivatives with respect to the flow parameter** ϵ). *Let* $\widetilde{x}^\mu = F_\epsilon(x)$ *be the change of spacetime coordinates defined in (2.0.16), and let* $\widetilde{\phi}$, $\widetilde{\nabla}_\mu \widetilde{\phi}$, $\widetilde{m}_{\mu\nu}$, $(\widetilde{m}^{-1})^{\mu\nu}$ *be the transformed fields defined in Definition 2.0.6. Then the following identities hold for all spacetime points* $\widetilde{x}$:

$$(2.0.31a) \qquad \partial_\epsilon\Big|_{\epsilon=0} \widetilde{\phi}|_{\widetilde{x}} = -Y^\alpha|_{\widetilde{x}} \nabla_\alpha \phi|_{\widetilde{x}},$$

$$(2.0.31b) \qquad \partial_\epsilon\Big|_{\epsilon=0} \widetilde{\nabla}_\mu \widetilde{\phi}|_{\widetilde{x}} = -\nabla_\mu Y^\alpha|_{\widetilde{x}} \nabla_\alpha \phi|_{\widetilde{x}} - Y^\alpha|_{\widetilde{x}} \nabla_\alpha \nabla_\mu \phi|_{\widetilde{x}} = -\nabla_\mu (Y^\alpha \nabla_\alpha \phi)|_{\widetilde{x}},$$

$$(2.0.31c) \qquad \partial_\epsilon\Big|_{\epsilon=0} \widetilde{m}_{\mu\nu}|_{\widetilde{x}} = -m_{\nu\alpha}|_{\widetilde{x}} \nabla_\mu Y^\alpha|_{\widetilde{x}} - m_{\mu\alpha}|_{\widetilde{x}} \nabla_\nu Y^\alpha|_{\widetilde{x}} - \underbrace{Y^\alpha|_{\widetilde{x}} \nabla_\alpha m_{\mu\nu}|_{\widetilde{x}}}_{0 \ for \ the \ Minkowski \ metric},$$

$$(2.0.31d) \quad \partial_\epsilon\Big|_{\epsilon=0} (\widetilde{m}^{-1})^{\mu\nu}|_{\widetilde{x}} = (m^{-1})^{\alpha\nu}|_{\widetilde{x}} \nabla_\alpha Y^\mu|_{\widetilde{x}} + (m^{-1})^{\mu\alpha}|_{\widetilde{x}} \nabla_\alpha Y^\nu|_{\widetilde{x}} - \underbrace{Y^\alpha|_{\widetilde{x}} \nabla_\alpha (m^{-1})^{\mu\nu}|_{\widetilde{x}}}_{0 \ for \ the \ Minkowski \ metric},$$

$$(2.0.31e) \quad \partial_\epsilon|_{\epsilon=0} \det M^{-1}|_{\widetilde{x}} = -\nabla_\alpha Y^\alpha|_{\widetilde{x}}.$$

Above and for the remainder of these notes, ∂_ϵ *denotes the derivative of an* $\epsilon-$*dependent quantity with the new coordinates* $\widetilde{x}$ *held fixed.*

Remark 2.0.3. In the language of differential geometry, the tilded fields are the *Lie derivatives* of the un-tilded fields with respect to the vectorfield $-Y$.

Proof. Recall that $\widetilde{x}^\mu = F_\epsilon^\mu(x)$, $x^\mu = F_{-\epsilon}^\mu(\widetilde{x})$, $F_0^\mu(x) = x^\mu$ (so that $x = \widetilde{x}$ when $\epsilon = 0$) and $\partial_\epsilon F_\epsilon^\mu(\cdot) = Y^\mu(\cdot)$. Therefore, using the chain rule, we compute that

$$(2.0.32) \qquad\qquad \partial_\epsilon|_{\epsilon=0} \widetilde{\phi}(\widetilde{x}) \overset{\mathrm{def}}{=} \partial_\epsilon|_{\epsilon=0} \phi(F_{-\epsilon}(\widetilde{x})) = \nabla_\alpha \phi|_{\widetilde{x}} \partial_\epsilon|_{\epsilon=0} F_{-\epsilon}^\alpha(\widetilde{x})$$
$$= -Y^\alpha|_{\widetilde{x}} \nabla_\alpha \phi|_{\widetilde{x}},$$

We have thus shown (2.0.31a).

Similarly, with the help of (2.0.21), and noting that $(M^{-1})_\mu^\nu = \delta_\mu^\nu$ when $\epsilon = 0$ and $\partial_\epsilon|_{\epsilon=0}\big[(M^{-1})_\mu^\alpha \circ F_{-\epsilon}\big]|_{\widetilde{x}} = -\nabla_\mu Y^\alpha|_{\widetilde{x}}$, we compute that

$(2.0.33)$
$$\partial_\epsilon|_{\epsilon=0} \widetilde{\nabla}_\mu \widetilde{\phi}(\widetilde{x}) \overset{\mathrm{def}}{=} \partial_\epsilon|_{\epsilon=0} \Big\{ (M^{-1})_\mu^\alpha \circ F_{-\epsilon}(\widetilde{x}) (\nabla_\alpha \phi) \circ F_{-\epsilon}(\widetilde{x}) \Big\}$$
$$= \Big\{ \partial_\epsilon|_{\epsilon=0} \big[(M^{-1})_\mu^\alpha \circ F_{-\epsilon}(\widetilde{x}) \big] \Big\} (\nabla_\alpha \phi) \circ F_{-\epsilon}(\widetilde{x}) + (M^{-1})_\mu^\alpha \circ F_{-\epsilon}(\widetilde{x}) \partial_\epsilon|_{\epsilon=0} (\nabla_\alpha \phi) \circ F_{-\epsilon}(\widetilde{x})$$
$$= -\nabla_\mu Y^\alpha|_{\widetilde{x}} \nabla_\alpha \phi|_{\widetilde{x}} - \underbrace{(M^{-1})_\mu^\alpha|_{\widetilde{x}}}_{\delta_\mu^\alpha} Y^\beta|_{\widetilde{x}} \nabla_\beta \nabla_\alpha \phi|_{\widetilde{x}}.$$

We have thus shown we have thus shown (2.0.31b). The proofs of (2.0.31c) and (2.0.31d) are similar, and we omit the details.

To prove (2.0.31e), we simply differentiate the expansion (2.0.22) with respect to ϵ and set $\epsilon = 0$. □

We now state the following simple corollary to Proposition 2.0.2.

Corollary 2.0.3 (The derivative of $\mathcal{L}$ with respect to the flow parameter ϵ). *Let $\mathcal{L}(\phi, \nabla\phi, m)$ be a C^2 Lagrangian. Then under the assumptions of Proposition 2.0.2, the following identity holds at all spacetime points:*

$$
(2.0.34) \qquad \partial_\epsilon|_{\epsilon=0} \mathcal{L}(\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m}) = -\frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\nabla\phi} Y^\alpha \nabla_\alpha \phi
$$
$$
-\frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\partial(\nabla_\mu\phi)} \nabla_\mu(Y^\alpha \nabla_\alpha \phi)
$$
$$
-\frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\partial m_{\mu\nu}} \Big\{ m_{\alpha\nu}\nabla_\mu Y^\alpha + m_{\mu\alpha}\nabla_\nu Y^\alpha + \underbrace{Y^\alpha \nabla_\alpha m_{\mu\nu}}_{0} \Big\}.
$$

Proof. By the chain rule, we have that

$$
(2.0.35) \qquad \partial_\epsilon \mathcal{L}(\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m}) = \frac{\partial\mathcal{L}(\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m})}{\partial\phi} \partial_\epsilon \widetilde{\phi}
$$
$$
+ \frac{\partial\mathcal{L}(\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m})}{\partial(\nabla_\mu\phi)} \partial_\epsilon \widetilde{\nabla}\widetilde{\phi} + \frac{\partial\mathcal{L}(\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m})}{\partial m_{\mu\nu}} \partial_\epsilon \widetilde{m}_{\mu\nu}.
$$

The relation (2.0.34) now follows from Proposition 2.0.2 and (2.0.35). $\square$

3. THE ENERGY-MOMENTUM TENSOR

The main goal of this section is to show that for a certain class of coordinate invariant Lagrangians $\mathcal{L}$, there exists an energy-momentum tensor $T^{\mu\nu}$. This $T^{\mu\nu}$ plays the same role in the analysis of the corresponding Euler-Lagrange equation corresponding to $\mathcal{L}$ as it did in our previous analysis of the linear wave equation. More precisely, for solutions to the Euler-Lagrange equation corresponding to $\mathcal{L}$, we will show that $\nabla_\mu T^{\mu\nu} = 0$. As we saw earlier in the course, this identity forms the basis for the derivation of conserved quantities in solutions to the Euler-Lagrange equations.

Theorem 3.1 (Derivation and divergence-free property of the energy-momentum tensor). *Let $\mathcal{L}(\phi, \nabla\phi, m)$ be a coordinate invariant Lagrangian (in the sense of Definition 2.0.7) that depends only on $\phi, \nabla\phi$, and the Minkowski metric m. Let*

$$
(3.0.36) \qquad\qquad T^{\mu\nu} \stackrel{def}{=} 2\frac{\partial\mathcal{L}}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu}\mathcal{L}
$$

be the energy-momentum tensor corresponding to $\mathcal{L}$. Then $T^{\mu\nu}$ is symmetric:

$$
(3.0.37) \qquad\qquad T^{\mu\nu} = T^{\nu\mu}, \qquad 0 \le \mu, \nu \le n.
$$

Furthermore, if ϕ verifies the Euler-Lagrange equation (1.0.7), the following divergence identity is verified by $T^{\mu\nu}$:

$$
(3.0.38) \qquad\qquad \nabla_\mu T^{\mu\nu} = 0, \qquad (\nu = 0, 1, 2, \cdots, n).
$$

Proof. The relation (3.0.37) follows easily from (3.0.36) since $m_{\mu\nu} = m_{\nu\mu}$.

We will now prove (3.0.38). To this end, let $\mathfrak{K} \subset \mathbb{R}^{1+n}$ be a compact spacetime subset, and let $Y : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be a smooth vectorfield with support contained in $\mathfrak{K}$. Let $\widetilde{x}$ be the change of variables (2.0.16), and consider the transformed quantities $\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m}, \widetilde{m}^{-1}$ given in Definition 2.0.6. Now by assumption, we have that $\mathcal{L}(\phi, \nabla\phi, m) = \mathcal{L}(\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m})$. Furthermore, by the standard change of variables theorem from advanced calculus, we have that $d^{1+n}x = \det\frac{\partial x}{\partial \widetilde{x}}d^{1+n}\widetilde{x} = \det M^{-1}d^{1+n}\widetilde{x}$, where the matrix M is defined in (2.0.18). Therefore, we have that

$$(3.0.39) \qquad \mathcal{A}[\phi; \mathfrak{K}] = \int_{\mathfrak{K}} \mathcal{L}(\phi, \nabla\phi, m)\, d^{1+n}x$$

$$= \int_{\mathfrak{K}} \mathcal{L}(\widetilde{\phi}, \widetilde{\nabla}\widetilde{\phi}, \widetilde{m}) \det M^{-1}d^{1+n}\widetilde{x}.$$

Now the left-hand side of (3.0.39) doesn't depend on ϵ. We therefore have that

$$(3.0.40) \qquad \frac{d}{d\epsilon}\big|_{\epsilon=0}\mathcal{A}[\phi; \mathfrak{K}] = 0.$$

On the other hand, we can differentiate under the integral on the right-hand side of (3.0.39) with respect to ϵ at $\epsilon = 0$ and use (2.0.31e) plus Corollary 2.0.3 (together with the fact that $x = \widetilde{x}$ when $\epsilon = 0$) to deduce that

$$(3.0.41) \qquad \frac{d}{d\epsilon}\big|_{\epsilon=0}\mathcal{A}[\phi; \mathfrak{K}] = \int_{\mathfrak{K}} -\frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\nabla\phi}Y^{\alpha}\nabla_{\alpha}\phi - \frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\partial(\nabla_{\mu}\phi)}\nabla_{\mu}(Y^{\alpha}\nabla_{\alpha}\phi)\, d^{1+n}x$$

$$- \int_{\mathfrak{K}} \frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\partial m_{\mu\nu}}\Big\{m_{\alpha\nu}\nabla_{\mu}Y^{\alpha} + m_{\mu\alpha}\nabla_{\nu}Y^{\alpha} + \underbrace{Y^{\alpha}\nabla_{\alpha}m_{\mu\nu}}_{0}\Big\}\, d^{1+n}x$$

$$- \int_{\mathfrak{K}} \mathcal{L}(\phi, \nabla\phi, m)\nabla_{\alpha}Y^{\alpha}\, d^{1+n}x.$$

Integrating by parts in (3.0.41) and using (3.0.40), we have that

$$(3.0.42) \qquad 0 = -\int_{\mathfrak{K}} \left\{\frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\nabla\phi} - \nabla_{\mu}\left(\frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\partial(\nabla_{\mu}\phi)}\right)\right\}Y^{\alpha}\nabla_{\alpha}\phi\, d^{1+n}x$$

$$- \int_{\mathfrak{K}} \frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\partial m_{\mu\nu}}\Big\{m_{\alpha\nu}\nabla_{\mu}Y^{\alpha} + m_{\mu\alpha}\nabla_{\nu}Y^{\alpha} + \underbrace{Y^{\alpha}\nabla_{\alpha}m_{\mu\nu}}_{0}\Big\}\, d^{1+n}x$$

$$- \int_{\mathfrak{K}} \mathcal{L}(\phi, \nabla\phi, m)\nabla_{\alpha}Y^{\alpha}\, d^{1+n}x.$$

We now note that the Euler-Lagrange equation (1.0.7) implies that the first line on the right-hand side of (3.0.42) is 0. Therefore, we collect the remaining terms together to derive that

$$(3.0.43) \qquad 0 = -\int_{\mathfrak{K}} \left\{\frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\partial m_{\mu\nu}} + \frac{1}{2}(m^{-1})^{\mu\nu}\mathcal{L}(\phi, \nabla\phi, m)\right\}\{m_{\alpha\nu}\nabla_{\mu}Y^{\alpha} + m_{\mu\alpha}\nabla_{\nu}Y^{\alpha}\}\, d^{1+n}x$$

$$= -\int_{\mathfrak{K}} \left\{2\frac{\partial\mathcal{L}(\phi, \nabla\phi, m)}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu}\mathcal{L}(\phi, \nabla\phi, m)\right\}m_{\alpha\nu}\nabla_{\mu}Y^{\alpha}\, d^{1+n}x.$$

Integrating by parts in (3.0.43), we deduce that

$$(3.0.44) \qquad 0 = \int_{\mathfrak{K}} \nabla_\mu \left\{ 2\frac{\partial \mathcal{L}(\phi, \nabla\phi, m)}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu}\mathcal{L}(\phi, \nabla\phi, m) \right\} m_{\alpha\nu} Y^\alpha \, d^{1+n}x.$$

Since (3.0.43) must hold for *all* such smooth vectorfields Y with support contained in $\mathfrak{K}$, we conclude that the divergence of the term in braces is 0:

$$(3.0.45) \qquad \nabla_\mu \left\{ 2\frac{\partial \mathcal{L}(\phi, \nabla\phi, m)}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu}\mathcal{L}(\phi, \nabla\phi, m) \right\} = 0.$$

We have thus shown that (3.0.38) holds. $\square$

Example 3.0.4. The Lagrangian for the linear wave equation is $\mathcal{L} = -\frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$. We therefore appeal to (3.0.36) and calculate that

$$(3.0.46) \qquad T^{\mu\nu} \stackrel{\text{def}}{=} 2\frac{\partial \mathcal{L}}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu}\mathcal{L}$$

$$= (m^{-1})^{\mu\alpha}(m^{-1})^{\nu\beta}\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi.$$

Remark 3.0.4. To derive (3.0.46), we have used the fact that if q is any quantity, and m is a symmetric invertible $(1+n) \times (1+n)$ matrix that depends on q, then

$$(3.0.47) \qquad \frac{d}{dq}(m^{-1})^{\mu\nu} = -(m^{-1})^{\mu\alpha}(m^{-1})^{\nu\beta}\frac{d}{dq}m_{\alpha\beta}, \qquad 0 \leq \mu, \nu \leq n.$$

You will derive the simple relation (3.0.47) in your homework. In particular, it follows from (3.0.47) that

$$(3.0.48) \qquad \frac{\partial(m^{-1})^{\mu\nu}}{\partial m_{\kappa\lambda}} = -(m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}.$$

On the left-hand side of (3.0.48), we are viewing the components $(m^{-1})^{\mu\nu}$ as functions of the components $m_{\kappa\lambda}$, $0 \leq \kappa, \lambda \leq n$.

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