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18.112 Functions of a Complex Variable
Fall 2008

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Lecture 6: Conformal Maps; Linear Transformations

(Text 69-80)

Remarks on Lecture 6

Problem 6 on p.83

Let C, D have center a, C', D' their images under mapping S. Then lines $\perp C$ and D go to circles $\perp C'$ and D' and these must be lines through the common center b of C' and D'.

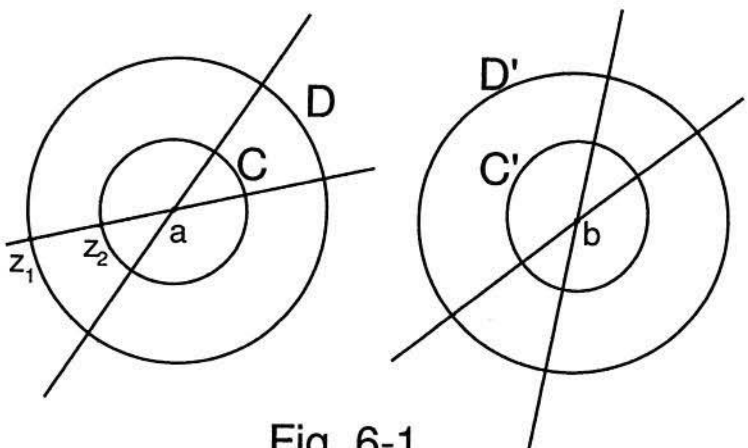

Fig. 6-1

In the extended plane, lines intersect always at ∞. Thus under S, a and ∞ go to b and ∞ or ∞ and b. Let

$$w_1 = Sz_1, \quad w_2 = Sz_2,$$

then

$$(z_1, z_2, a, \infty) = \begin{cases} (w_1, w_2, b, \infty), \\ (w_1, w_2, \infty, b), \end{cases}$$

and

$$\left| \frac{z_1 - a}{z_2 - a} : \frac{z_1 - \infty}{z_2 - \infty} \right| = \begin{cases} \left| \frac{w_1 - b}{w_2 - b} : \frac{w_1 - \infty}{w_2 - \infty} \right|, \\ \\ \left| \frac{w_1 - \infty}{w_2 - \infty} : \frac{w_1 - b}{w_2 - b} \right|. \end{cases}$$

So

$$\frac{r}{R} = \begin{cases} \frac{r_1}{R_1}, \\ \\ \frac{R_1}{r_1}. \end{cases}$$

Theorem 1 *(Implying Problem 7, p. 83 and Problem 6, p. 88.)*

If A and B are two nonintersecting circles there exists a linear transformation mapping A and B into concentric circles.

Proof. First transform A to a line A_1. This sends B to a circle B_1. Consider the line ℓ from the center of B perpendicular to A_1. Let M be the point $\ell \cap A_1$. With M as center construct the circle C cutting B_1 orthogonally. Then take a linear transformation sending one of the points in $\ell \cap C$ to ∞. It sends C and ℓ into orthogonal lines m and n. Then A_1 and B_1 are sent into circles A_2 and B_2 which cut m and n orthogonally and are therefore concentric.

Problem 5 on p.83

Suppose S maps a to 0. Since a and $\frac{R^2}{\bar{a}}$ are symmetric with respect to $|z| = R$, S maps $\frac{R^2}{\bar{a}}$ to ∞. The transformation

$$S_0(z) = R^2 \frac{z - a}{R^2 - \bar{a}z}$$

maps a to 0 and $\frac{R^2}{\bar{a}}$ to ∞, and maps $|z| = R$ into itself since

$$|Re^{i\theta} - a| = |R - \bar{a}e^{i\theta}|.$$

If T also has this property, then TS_0^{-1} maps 0 to 0 and maps ∞ to ∞, so

$$TS_0^{-1} = cz$$

with $|c| = 1$. Thus

$$T = R^2 e^{i\theta} \frac{z - a}{R^2 - \bar{a}z}.$$