Solution for 18.112 Final Examination

Problem 1.
Method 1 (Geometric way): Let A, B, C be points on the complex plane corresponding to complex numbers a, b, c. Then

$$\overrightarrow{AB} = b - a,\ \overrightarrow{BC} = c - b,\ \overrightarrow{CA} = a - c$$

and

$$\angle A = \arg\frac{b-a}{c-a},\ \angle B = \arg\frac{c-b}{a-b},\ \angle C = \arg\frac{a-c}{b-c}.$$

By the condition

$$\frac{b-a}{c-a} = \frac{a-c}{b-c}$$

we get

$$\frac{|\overrightarrow{AB}|}{|\overrightarrow{CA}|} = \frac{|\overrightarrow{CA}|}{|\overrightarrow{BC}|}$$

and

$$\angle A = \angle C,$$

i.e.

$$|\overrightarrow{AB}||\overrightarrow{BC}| = |\overrightarrow{CA}||\overrightarrow{CA}|$$

and

$$|\overrightarrow{BC}| = |\overrightarrow{AB}|.$$

So

$$|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|,$$

i.e.

$$|b - a| = |c - a| = |b - c|.$$

Method 2 (Algebraic way): First note that

$$\frac{b-a}{c-a} = \frac{a-c}{b-c} \Longrightarrow \frac{b-a}{c-a} = \frac{a-c}{b-c} = \frac{b-a+a-c}{c-a+b-c} = \frac{b-c}{b-a}.$$

But

$$\left|\frac{b-a}{c-a} \cdot \frac{a-c}{b-c} \cdot \frac{b-c}{b-a}\right| = 1,$$

So

$$\left|\frac{b-a}{c-a}\right| = \left|\frac{a-c}{b-c}\right| = \left|\frac{b-c}{b-a}\right| = 1,$$

i.e.

$$|b-a| = |c-a| = |b-c|.$$

Problem 2.

Solution: By rewriting the series as

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}} = \sum_{n=1}^{\infty} \frac{1}{z^n + z^{-n}},$$

we can see that it converges in $|z| > 1$ and $|z| < 1$.

On $|z| = 1$, we can write $z = e^{i\theta}$. Then

$$\frac{1}{z^n + z^{-n}} = \frac{1}{e^{in\theta} + e^{-in\theta}} = \frac{1}{\cos n\theta} \nrightarrow 0$$

as $n \to \infty$ for any θ, so the series does not converge on $|z| = 1$.

Moreover, for any compact subset K of $|z| > 1$ or $|z| < 1$, we can find some constant $C > 1$ or $C < 1$ such that $|z| > C > 1$ or $|z| < C < 1$ on K. Thus

$$\left|\frac{1}{z^n + z^{-n}}\right| < \frac{1}{C^n + C^{-n}}$$

for all $z \in K$. Since

$$\sum_{n=1}^{\infty} \frac{1}{C^n + C^{-n}}$$

will always converge, we know that

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$$

converges uniformly on every compact subset of $|z| > 1$ and $|z| < 1$. So by the equivalent form of Weierstrass theorem on page 177, the sum $f(z)$ is holomorphic in $|z| > 1$ and $|z| < 1$.

Problem 3.

Solution: Note that $|z| = 2$ on γ, we have

$$\int_\gamma \frac{|z|e^z}{z^2}dz = \int_\gamma \frac{2e^z}{z^2}dz.$$

Now the function $\frac{2e^z}{z^2}$ has only one pole at $z = 0$, and by Taylor expansion,

$$\begin{aligned}\frac{2e^z}{z^2} &= 2\frac{1 + z + \frac{z^2}{2!} + \cdots}{z^2} \\ &= 2\left(\frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \cdots\right),\end{aligned}$$

So

$$\mathrm{Res}_{z=0}\frac{2e^z}{z^2} = 2.$$

By Residue theorem,

$$\int_\gamma \frac{|z|e^z}{z^2}dz = 2 \cdot 2\pi i = 4\pi i.$$

Problem 4.

Solution: We can write

$$f(z) = (z - z_0)^{-h}g(z),$$

where $g(z)$ is holomorphic near z_0. Then by (24) on page 120,

$$\begin{aligned}g^{(h-1)}(z_0) &= \frac{(h-1)!}{2\pi i}\int_C \frac{g(z)}{(z - z_0)^h}dz \\ &= (h-1)!\mathrm{Res}_{z=z_0}f(z).\end{aligned}$$

So

$$\mathrm{Res}_{z=z_0}f(z) = \frac{1}{(h-1)!}\left\{\frac{d^{h-1}}{dz^{h-1}}(z - z_0)^h f(z)\right\}_{z=z_0}.$$

Problem 5.

Solution: In $|z|<1$, by using geometric series for $\frac{1}{1-z}$ and $\frac{1}{1-z/2}$, we have

$$\begin{aligned}
\frac{1}{(z-1)(z-2)} &= \frac{1}{1-z}\frac{\frac{1}{2}}{1-\frac{z}{2}} \\
&= (1+z+z^2+\cdots)\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right) \\
&= \frac{1}{2}\Bigg[1+\left(1+\frac{1}{2}\right)z+\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2\right)z^2+\cdots \\
&\qquad +\left(1+\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^n\right)z^n+\cdots\Bigg] \\
&= \frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\cdots+\left(1-\frac{1}{2^{n+1}}\right)z^n+\cdots.
\end{aligned}$$

By the same way, in $|z|>2$, use the geometric series for $\frac{1}{1-1/z}$ and $\frac{1}{1-2/z}$, we get

$$\begin{aligned}
\frac{1}{(z-1)(z-2)} &= \frac{1}{z^2}\frac{1}{1-\frac{1}{z}}\frac{1}{1-\frac{2}{z}} \\
&= \frac{1}{z^2}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right)\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right) \\
&= \frac{1}{z^2}\left[1+(1+2)\frac{1}{z}+\left(1+2+2^2\right)\frac{1}{z^2}+\cdots+(1+2+\cdots+2^n)\frac{1}{z^n}+\cdots\right] \\
&= \frac{1}{z^2}+3\frac{1}{z^3}+7\frac{1}{z^4}+\cdots+(2^{n+1}-1)\frac{1}{z^{n+2}}+\cdots.
\end{aligned}$$

Problem 6.

Solution: Let

$$g(z)=f(z)-z,\ h(z)=-z,$$

both are analytic in $|z|\le 1$. On the boundary $|z|=1$, we have

$$|g(z)-h(z)|=|f(z)|<1=|h(z)|,$$

thus by Rouche Theorem, $g(z)=f(z)-z$ has exactly one zero inside $|z|=1$.