# Lecture 35

Before moving on to integration, we make a few more remarks about orientations.

Let $X, Y$ be oriented manifolds. A diffeomorphism $f : X \to Y$ is orientation preserving if for every $p \in X$, the map

$$df_p : T_pX \to T_qY \tag{6.85}$$

is orientation preserving, where $q = f(p)$.

Let $V$ be open in $X$, let $U$ be open in $\mathbb{R}^n$, and let $\phi : U \to V$ be a parameterization.

**Definition 6.32.** The map $\phi$ is an *oriented parameterization* if it is orientation preserving.

Suppose $\phi$ is orientation reversing. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by

$$A(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n). \tag{6.86}$$

The map $A$ is orientation reversing. Let $U' = A^{-1}(U)$, and define $\phi' = \phi \circ A : U' \to V$. Both $\phi$ and $A$ are orientation reversing, so $\phi'$ is orientation preserving.

Thus, for every point $p \in X$, there exists an oriented parameterization of $X$ at $p$.

## 6.7 Integration on Manifolds

Our goal for today is to take any $\omega \in \Omega^n_c(X)$ and define

$$\int_X \omega. \tag{6.87}$$

First, we consider a special case:

Let $\phi : U \to V$ be an oriented parameterization. Let $U$ be open in $\mathbb{R}^n$, and let $V$ be open in $X$. Take any $\omega \in \Omega^n_c(V)$. Then

$$\int_V \omega = \int_U \phi^*\omega, \tag{6.88}$$

where $\phi^*\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$, where $f \in \mathcal{C}^\infty_0(U)$ and

$$\int_U \phi^*\omega = \int_U f. \tag{6.89}$$

**Claim.** *The above definition for $\int \omega$ does not depend on the choice of oriented parameterization $\phi$.*

*Proof.* Let $\phi_i : U_i \to V,\ i = 1, 2$, be oriented parameterizations. Let $\omega \in \Omega_c^n(V_1 \cap V_2)$. Define

$$\begin{aligned} U_{1,2} &= \phi_1^{-1}(V_1 \cap V_2), && (6.90)\\ U_{2,1} &= \phi_2^{-1}(V_1 \cap V_2), && (6.91) \end{aligned}$$

which are open sets in $\mathbb{R}^n$.

Both $\phi_1$ and $\phi_2$ are diffeomorphisms, and we have the diagram

$$\begin{array}{ccc} V_1 \cap V_2 & = & V_1 \cap V_2 \\ {\scriptstyle \phi_1}\uparrow & & {\scriptstyle \phi_2}\uparrow \\ U_{1,2} & \xrightarrow{f} & U_{2,1}. \end{array} \qquad (6.92)$$

Therefore, $f = \phi_2^{-1} \circ \phi_1$ is a diffeomorphism, and $\phi_1 = \phi_2 \circ f$. Integrating,

$$\begin{aligned} \int_{U_1} \phi_1^* \omega &= \int_{U_{1,2}} \phi_1^* \omega \\ &= \int_{U_{1,2}} (\phi_2 \circ f)^* \omega \\ &= \int_{U_{1,2}} f^*(\phi_2^* \omega). \end{aligned} \qquad (6.93)$$

Note that $f$ is orientation preserving, because $\phi_1$ and $\phi_2$ are orientation preserving.

Using the change of variables formula,

$$\begin{aligned} \int_{U_{1,2}} f^* \phi_2^* \omega &= \int_{U_{2,1}} \phi_2^* \omega \\ &= \int_{U_2} \phi_2^* \omega. \end{aligned} \qquad (6.94)$$

So, for all $\omega \in \Omega_c^n(V_1 \cap V_2)$,

$$\int_{V_1} \omega = \int_{U_1} \phi_1^* \omega = \int_{U_2} \phi_2^* \omega = \int_{V_2} \omega. \qquad (6.95)$$

□

Above, we showed above how to take integrals over open sets, and now we generalize.

To define the integral, we need the following two inputs:

1. a set of oriented parameterizations $\phi_i : U_i \to V_i,\ i = 1, 2, \dots$, such that $X = \bigcup V_i$,

2. a partition of unity $\rho_i \in \mathcal{C}_0^\infty(V_i)$ subordinate to the cover $\{V_i\}$.

**Definition 6.33.** Let $\omega \in \Omega_c^n(X)$. We define the integral

$$\int_X \omega = \sum_{i=1}^\infty \int_{V_i} \rho_i \omega. \tag{6.96}$$

One can check various standard properties of integrals, such as linearity:

$$\int_X \omega_1 + \omega_2 = \int_X \omega_1 + \int_X \omega_2. \tag{6.97}$$

We now show that this definition is independent of the choice of the two inputs (the parameterizations and the partition of unity).

Consider two different inputs:

1. oriented parameterizations $\phi_j' : U_j' \to V_j'$, $j = 1, 2, \dots$, such that $X = \bigcup V_j'$,
2. a partition of unity $\rho_i' \in \mathcal{C}_0^\infty(V_j')$ subordinate to the cover $\{V_j'\}$.

Then,

$$\begin{aligned} \int_{V_i} \rho_i \omega &= \int_{V_i} \left( \sum_{j=1}^\infty \rho_j' \omega \right) \\ &= \sum_{j=1}^\infty \int_{V_i} \rho_i \rho_j' \omega \\ &= \sum_{j=1}^\infty \int_{V_i \cap V_j'} \rho_i \rho_j' \omega. \end{aligned} \tag{6.98}$$

Summing over $i$,

$$\begin{aligned} \sum_i \int_{V_i} \rho_i \omega &= \sum_{i,j=1}^\infty \int_{V_i \cap V_j'} \rho_i \rho_j' \omega \\ &= \sum_j \int_{V_j'} \rho_j' \omega, \end{aligned} \tag{6.99}$$

where the first term equals the last term by symmetry. Therefore, the integral $\int \omega$ is independent of the choices of these two inputs.

Let $X \subseteq \mathbb{R}^N$ be an oriented connected $n$-dimensional manifold.

**Theorem 6.34.** *For any $\omega \in \Omega_c^n(X)$, the following are equivalent:*

1. $\int_X \omega = 0$,

*2.* $\omega \in d\Omega_c^{n-1}(X)$.

*Proof.* This will be a five step proof:

Step 1: The following lemma is called the Connectivity Lemma.

**Lemma 6.35.** *Given $p, q \in X$, there exists open sets $W_j$, $j = 0, \dots, N+1$, such that each $W_j$ is diffeomorphic to an open set in $\mathbb{R}^n$, and such that $p \in W_0$, $q \in W_{N+1}$, and $W_i \cap W_{i+1} \neq \phi$.*

*Proof Idea:* Fix $p$. The points $q$ for which this is true form an open set. The points $q$ for which this isn't true also form an open set. Since $X$ is connected, only one of these sets is in $X$. □

Step 2: Let $\omega_1, \omega_2 \in \Omega_c^n(X)$. We say that $\omega_1 \sim \omega_2$ if

$$\int_X \omega_1 = \int_X \omega_2. \tag{6.100}$$

We can restate the theorem as

$$\omega_1 \sim \omega_2 \iff \omega_1 - \omega_2 \in d\Omega_c^{n-1}(X). \tag{6.101}$$

Step 3: It suffices to prove the statement (6.101) for $\omega_1 \in \Omega_c^n(V)$ and $\omega_2 \in \Omega_c^n(V')$, where $V, V'$ are diffeomorphic to open sets in $\mathbb{R}^n$.

Step 4: We use a partition of unity

**Lemma 6.36.** *The theorem is true if $V = V'$.*

*Proof.* Let $\phi : U \to V$ be an orientation preserving parameterization. If $\omega_1 \sim \omega_2$, then

$$\int \phi^*\omega_1 = \int \phi^*\omega_2, \tag{6.102}$$

which is the same as saying that

$$\phi^*\omega_1 - \phi^*\omega_2 \in d\Omega_c^{n-1}(U), \tag{6.103}$$

which is the same as saying that

$$\omega_1 - \omega_2 \in d\Omega_c^{n-1}(V). \tag{6.104}$$

□

Step 5: In general, by the Connectivity Lemma, there exists sets $W_i$, $i = 0, \dots, N+1$, such that each $W_i$ is diffeomorphic to an open set in $\mathbb{R}^n$. We can choose $W_0 = V$ and $W_{N+1} = V'$ and $W_i \cap W_{i+1} \neq \phi$ (where $\phi$ here is the empty set).

We can choose $\mu_i \in \Omega_c^n(W_i \cap W_{i+1})$ such that

$$c = \int_V \omega_1 = \int \mu_0 = \cdots = \int \mu_{N+1} = \int_{V'} \omega_2. \tag{6.105}$$

So,

$$\omega_1 \sim \mu_0 \sim \cdots \sim \mu_N \sim \omega_2. \tag{6.106}$$

We know that $\mu_0 - \omega_1 \in d\Omega_c^{n-1}$ and $\omega_2 - \mu_{N+1} \in d\Omega_C^{n-1}$ Also, each difference $\omega_i - \omega_{i+1} \in d\Omega_c^{n-1}$. Therefore, $\omega_1 - \omega_2 \in d\Omega_c^{n-1}$. □

## 6.8 Degree on Manifolds

Suppose that $X_1, X_2$ are oriented $n$-dimensional manifolds, and let $f : X_1 \to X_2$ be a proper map (that is, for every compact set $A \subseteq X$, the set pre-image $f^{-1}(A)$ is compact). It follows that if $\omega \in \Omega_c^k(X_2)$, then $f^*\omega \in \Omega_c^k(X_1)$.

**Theorem 6.37.** *If $X_1, X_2$ are connected and $f : X_1 \to X_2'$ is a proper $\mathcal{C}^\infty$ map, then there exists a topological invariant of $f$ (called the* degree of $f$*) written* $\deg(f)$ *such that for every $\omega \in \Omega_c^n(X_2)$,*

$$\int_{X_1} f^*\omega = \deg(f) \int_{X_2} \omega. \tag{6.107}$$

*Proof.* The proof is pretty much verbatim of the proof in Euclidean space. □

Let us look at a special case. Let $\phi_1 : U \to V$ be an oriented parameterization, and let $V_1$ be open in $X_1$. Let $f : X_1 \to X_2$ be an oriented diffeomorphism. Define $\phi_2 = f \circ \phi_1$, which is of the form $\phi_2 : U \to V_2$, where $V_2 = f(V_1)$. Notice that $\phi_2$ is an oriented parameterization of $V_2$.

Take $\omega \in \Omega_c^n(V_2)$ and compute the integral

$$\begin{aligned} \int_{V_1} f^*\omega &= \int_U \phi_1^* f^*\omega \\ &= \int_U (f \circ \phi_1)^*\omega \\ &= \int_U \phi_2^*\omega. \end{aligned} \tag{6.108}$$

The $n$-form $\omega$ is compactly supported on $V_2$, so

$$\begin{aligned} \int_{V_1} f^*\omega &= \int_U \phi_2^*\omega \\ &= \int_{X_2} \omega. \end{aligned} \tag{6.109}$$

On the other hand,

$$\int_{X_1} f^*\omega = \int_{V_1} f^*\omega. \tag{6.110}$$

Combining these results,

$$\int_{X_1} f^*\omega = \int_{X_2} \omega. \tag{6.111}$$

Therefore,

$$\deg(f) = 1. \tag{6.112}$$

So, we have proved the following theorem, which is the Change of Variables theorem for manifolds:

**Theorem 6.38.** *Let $X_1, X_2$ be connected oriented $n$-dimensional manifolds, and let $f : X_1 \to X_2$ be an orientation preserving diffeomorphism. Then, for all $\omega \in \Omega^n_c(X_2)$,*

$$\int_{X_1} f^*\omega = \int_{X_2} \omega. \tag{6.113}$$