Lecture 28

Let U, V be connected open sets of $\mathbb{R}^n$, and let $f : U \to V$ be a diffeomorphism. Then

$$\deg(f) = \begin{cases} +1 & \text{if } f \text{ is orient. preserving,} \\ -1 & \text{if } f \text{ is orient. reversing.} \end{cases} \tag{5.124}$$

We showed that given any $\omega \in \Omega^n_c(V)$,

$$\int_U f^*\omega = \pm \int_V \omega. \tag{5.125}$$

Let $\omega = \phi(x) dx_1 \wedge \cdots \wedge dx_n$, where $\phi \in \mathcal{C}^\infty_0(V)$. Then

$$f^*\omega = \phi(f(x)) \det \left[\frac{\partial f_i}{\partial x_j}(x) \right] dx_1 \wedge \cdots \wedge dx_n, \tag{5.126}$$

so,

$$\int_U \phi(f(x)) \det \left[\frac{\partial f_i}{\partial x_j} \right] dx = \pm \int_V \phi(x) dx. \tag{5.127}$$

Notice that

$$f \text{ is orientation preserving} \iff \det \left[\frac{\partial f_i}{\partial x_j}(x) \right] > 0, \tag{5.128}$$

$$f \text{ is orientation reversing} \iff \det \left[\frac{\partial f_i}{\partial x_j}(x) \right] < 0. \tag{5.129}$$

So, in general,

$$\int_U \phi(f(x)) \left| \det \left[\frac{\partial f_i}{\partial x_j}(x) \right] \right| dx. \tag{5.130}$$

As usual, we assumed that $f \in \mathcal{C}^\infty$.

Remark. The above is true for $\phi \in \mathcal{C}^1_0$, a compactly supported continuous function. The proof of this is in section 5 of the Supplementary Notes. The theorem is true even if only $f \in \mathcal{C}^1$ (the notes prove it for $f \in \mathcal{C}^2$).

Today we show how to compute the degree in general.

Let U, V be connected open sets in $\mathbb{R}^n$, and let $f : U \to V$ be a proper $\mathcal{C}^\infty$ map.

Claim. *Let B be a compact subset of V, and let $A = f^{-1}(B)$. If U_0 is an open subset of U with $A \subseteq U_0$, then there exists an open subset V_0 of V with $B \subseteq V_0$ such that $f^{-1}(V_0) \subseteq U_0$.*

Proof. Let $C \subseteq V$ be a compact set with $B \subseteq \text{Int } C$, and let $W = f^{-1}(C) - U_0$. The set W is compact, so the set $f(W)$ is also compact. Moreover, $f(W) \cap B = \phi$ since $f^{-1}(B) \subseteq U_0$.

Now, let $V_0 = \text{Int } C - f(W)$. This set is open, and

$$\begin{aligned} f^{-1}(V_0) &\subseteq f^{-1}(\text{Int } C) - W \\ &\subseteq U_0. \end{aligned} \tag{5.131}$$

□

Claim. *If $X \subseteq U$ is closed, then $f(X)$ is closed in V.*

Proof. Take any point $p \in V - f(x)$. Then $f^{-1}(p) \in U - X$. Apply the previous result with $B = \{p\}$, $A = f^{-1}(p)$, and $U_0 = U - X$. There exists an open set $V_0 \ni p$ such that $f^{-1} \subseteq U - X$. The set $V_0 \cap f(X) = \phi$, so $V - f(X)$ is open in V. □

We now remind you of Sard's Theorem. Let $f : U \to V$ be a proper $\mathcal{C}^\infty$ map. We define the critical set

$$C_f = \{p \in U : Df(p) \text{ is not bijective}\}. \tag{5.132}$$

The set C_f is closed. The set $f(C_f)$ in V is a set of measure zero. The set $f(C_f)$ is closed as well, since f is proper.

Definition 5.18. A point $q \in V$ is a *regular value* of f if $q \in V - f(C_f)$.

Sard's Theorem basically says that there are "lots" of regular values.

Lemma 5.19. *If q is a regular value, then $f^{-1}(q)$ is a finite set.*

Proof. First, $p \in f^{-1}(q) \implies p \notin C_f$. So, $Df(p) : \mathbb{R}^n \to \mathbb{R}^n$ is bijective. By the IFT, the map f is a diffeomorphism of a neighborhood U_p of $p \in U$ onto a neighborhood of q. In particular, since f is one-to-one and onto,

$$U_p \cap f^{-1}(q) = \{p\}. \tag{5.133}$$

Consider the collection $\{U_p : p \in f^{-1}(q)\}$, which is an open cover of $f^{-1}(q)$. The H-B Theorem tells us that there exists a finite subcover $\{U_{p_i}, i = 1, \dots, N\}$. Hence,

$$f^{-1}(q) = \{p_1, \dots, p_N\}. \tag{5.134}$$

□

Theorem 5.20. *The degree of f is*

$$\deg(f) = \sum_{i=1}^{N} \sigma_{p_i}, \tag{5.135}$$

where

$$\sigma_{p_i} = \begin{cases} +1 & \text{if } Df(p_i) \text{ is orient. preserving,} \\ -1 & \text{if } Df(p_i) \text{ is orient. reversing.} \end{cases} \tag{5.136}$$

So, to calculate the degree, you just pick any regular value q and "count" the number of points in the pre-image of q, keeping track of the value of σ_{p_i}.

Proof. For each $p_i \in f^{-1}(q)$, let U_{p_i} be an open neighborhood of p_i such that f maps U_{p_i} diffeomorphically onto a neighborhood of q. We can assume that the U_{p_i}'s do not intersect.

Now, choose a neighborhood V_0 of q such that

$$f^{-1}(V_0) \subseteq \bigcup U_{p_i}. \tag{5.137}$$

Next, replace each U_{p_i} by $U_{p_i} \cap f^{-1}(V_0)$. So, we can assume the following:

1. f is a diffeomorphism of U_{p_i} onto V_0,
2. $f^{-1}(V_0) = \bigcup U_{p_i}$,
3. The U_{p_i}'s don't intersect.

Choose $\omega \in \Omega^n_c(V_0)$ such that

$$\int_V \omega = 1. \tag{5.138}$$

Then,

$$\begin{aligned} \int_U f^*\omega &= \sum_i \int_{U_{p_i}} f^*\omega \\ &= \sum_i \sigma_{p_i} \int_{V_0} \omega \\ &= \sum_i \sigma_{p_i}. \end{aligned} \tag{5.139}$$

But,

$$\int_U f^*\omega = (\deg f) \int_U \omega = \deg f, \tag{5.140}$$

so

$$\sum \sigma_{p_i} = \deg f. \tag{5.141}$$

□

This is a very nice theorem that is not often discussed in textbooks.

The following is a useful application of this theorem. Suppose $f^{-1}(q)$ is empty, so $q \notin f(U)$. Then $q \notin f(C_f)$, so q is a regular value. Therefore,

$$\deg(f) = 0. \tag{5.142}$$

This implies the following useful theorem.

Theorem 5.21. *If* $\deg(f) \neq 0$*, then* $f : U \to V$ *is onto.*

This theorem can let us know if a system of non-linear equations has a solution, simply by calculating the degree. The way to think about this is as follows. Let $f = (f_1, \dots, f_n)$ and let $q = (c_1, \dots, c_n) \in V$. If $q \in f(U)$ then there exists a solution $x \in U$ to the system of non-linear equations

$$f_i(x) = c_i, \quad i = 1, \dots, n. \tag{5.143}$$