Lecture 23

Let U be an open set in $\mathbb{R}^n$. For each $k = 0, \dots, n-1$, we define the differential operator

$$d : \Omega^k(U) \to \Omega^{k+1}(U). \tag{4.175}$$

These maps are the n basic vector calculus operations in n-dimensional calculus. We review how d is defined.

For $k = 0$, $\Omega^0(U) = \mathcal{C}^\infty(U)$. Let $f \in \mathcal{C}^\infty(U)$, and let $c = f(p)$, where $p \in U$. The mapping $df_p : T_p\mathbb{R}^n \to T_c\mathbb{R} = \mathbb{R}$ maps $T_p\mathbb{R}^n$ to $\mathbb{R}$, so $df_p \in T_p^*\mathbb{R}^n$. The map $df \in \Omega^1(U)$ is a one-form that maps $p \in U$ to $df_p \in T_p^*\mathbb{R}^n$. A formula for this in coordinates is

$$df = \sum \frac{\partial f}{\partial x_i} dx_i. \tag{4.176}$$

In k dimensions, d is a map

$$d : \Omega^k(U) \to \Omega^{k+1}(U). \tag{4.177}$$

Given $\omega \in \Omega^k(U)$, ω can be written uniquely as

$$\begin{aligned} \omega &= \sum_I a_I dx_I \\ &= \sum_I a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \end{aligned} \tag{4.178}$$

where $i_1 < \cdots < i_k$ and each $a_I \in \mathcal{C}^\infty(U)$. Then, we define

$$\begin{aligned} d\omega &= \sum da_I \wedge dx_I \\ &= \sum_{i,I} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I, \end{aligned} \tag{4.179}$$

where each I is strictly increasing.

The following are some basic properties of the differential operator d:

1. If $\mu \in \Omega^k(U)$ and $\nu \in \Omega^\ell(U)$, then

$$d\mu \wedge \nu = d\mu \wedge \nu + (-1)^k \mu \wedge d\nu. \tag{4.180}$$

2. For and $\omega \in \Omega^k(U)$,

$$d(d\omega) = 0. \tag{4.181}$$

Remark. Let I be any multi-index, and let $a_I \in \mathcal{C}^\infty(U)$. Then

$$d(a_I dx_I) = da_I \wedge dx_I. \tag{4.182}$$

We now prove the above two basic properties of the differential operator.

Claim. *If* $\mu \in \Omega^k(U)$ *and* $\nu \in \Omega^\ell(U)$*, then*

$$d\mu \wedge \nu = d\mu \wedge \nu + (-1)^k \mu \wedge d\nu. \tag{4.183}$$

Proof. Take $\mu = \sum a_I dx_I$ and $\nu = \sum b_J dx_J$, where I, J are strictly increasing. Then

$$\mu \wedge \nu = \sum a_I b_J \underbrace{dx_I \wedge dx_J}_{\text{no longer increasing}}. \tag{4.184}$$

Then

$$\begin{aligned} d(\mu \wedge \nu) &= \sum_{i,I,J} \frac{\partial a_I b_J}{\partial x_i} dx_i \wedge dx_I \wedge dx_J \\ &= \sum \frac{\partial a_I}{\partial x_i} b_J dx_i \wedge dx_I \wedge dx_J \qquad (I) \\ &\quad + \sum a_I \frac{\partial b_J}{\partial x_i} dx_i \wedge dx_I \wedge dx_J, \quad (II) \end{aligned} \tag{4.185}$$

We calculate sums (I) and (II) separately.

$$\begin{aligned} (I) &= \sum_{i,I,J} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I \wedge b_J dx_J \\ &= \left(\sum_{i,I} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I \right) \wedge \sum_J b_J dx_J \\ &= d\mu \wedge \nu. \end{aligned} \tag{4.186}$$

$$\begin{aligned} (II) &= \sum_{i,I,J} a_I \frac{\partial b_J}{\partial x_i} dx_i \wedge dx_I \wedge dx_J \\ &= (-1)^k \sum_{i,I,J} a_I dx_I \wedge \frac{\partial b_J}{\partial x_i} dx_i \wedge dx_J \\ &= (-1)^k \left(\sum_I a_I dx_I \right) \wedge \sum_{i,J} \frac{\partial b_J}{\partial x_i} dx_i \wedge dx_J \\ &= (-1)^k \mu \wedge d\nu. \end{aligned} \tag{4.187}$$

So,

$$\begin{aligned} d(\mu \wedge \nu) &= (I) + (II) \\ &= d\mu \wedge \nu + (-1)^k \mu \wedge d\nu. \end{aligned} \tag{4.188}$$

□

Claim. *For and $\omega \in \Omega^k(U)$,*

$$d(d\omega) = 0. \tag{4.189}$$

Proof. Let $\omega = \sum a_I dx_I$, so

$$d\omega = \sum_{j,I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I. \tag{4.190}$$

Then,

$$d(d\omega) = \sum_{i,j,I} \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I. \tag{4.191}$$

Note that if $i = j$, then there is a repeated term in the wedge product, so

$$d(d\omega) = \sum_{i<j} \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I \tag{4.192}$$

$$+ \sum_{i>j} \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I. \tag{4.193}$$

Note that $dx_i \wedge dx_j = -dx_j \wedge dx_i$. We relabel the second summand to obtain

$$\begin{aligned} d(d\omega) &= \sum_{i<j} \left(\underbrace{\frac{\partial^2 a_I}{\partial x_i \partial x_j} - \frac{\partial^2 a_I}{\partial x_j \partial x_i}}_{0} \right) dx_i \wedge dx_j \wedge dx_I \\ &= 0. \end{aligned} \tag{4.194}$$

□

Definition 4.42. A k-form $\omega \in \Omega^k(U)$ is *decomposable* if $\omega = \mu_1 \wedge \cdots \wedge \mu_k$, where each $\mu_i \in \Omega^1(U)$.

Theorem 4.43. *If ω is decomposable, then*

$$d\omega = \sum_{i=1}^{k} (-1)^{i-1} \mu_1 \wedge \cdots \wedge \mu_{i-1} \wedge d\mu_i \wedge \mu_{i+1} \wedge \cdots \wedge \mu_k. \tag{4.195}$$

Proof. The proof is by induction.

The case $k = 1$ is obvious. We show that if the theorem is true for $k-1$, then the theorem is true for k.

$$\begin{aligned}
d((\mu_1 \wedge \cdots \wedge \mu_{k-1}) \wedge \mu_k) &= (d(\mu_1 \wedge \cdots \wedge \mu_{k-1})) \wedge \mu_k \\
&\quad + (-1)^{k-1}(\mu_1 \wedge \cdots \wedge \mu_{k-1}) \wedge d\mu_k \\
&= \sum_{i=1}^{k-1} (-1)^{i-1} \mu_1 \wedge \cdots \wedge d\mu_i \wedge \cdots \wedge \mu_{k-1} \wedge \mu_k \\
&\quad + (-1)^{k-1}(\mu_1 \wedge \cdots \wedge \mu_{k-1} \wedge \mu_k) \\
&= \sum_{i=1}^{k} (-1)^{i-1} \mu_1 \wedge \cdots \wedge d\mu_i \wedge \cdots \wedge \mu_k.
\end{aligned} \tag{4.196}$$

□

4.10 Pullback Operation on Exterior Forms

Another important operation in the theory of exterior forms is the *pullback operator*. This operation is not introduced in 18.01 or 18.02, because vector calculus in not usually taught rigorously.

Let U be open in $\mathbb{R}^n$ and V be open in $\mathbb{R}^m$, and let $f : U \to V$ be a $\mathcal{C}^\infty$ map. We can write out in components $f = (f_1, \ldots, f_n)$, where each $f_i \in \mathcal{C}^\infty(U)$. Let $p \in U$ and $q = f(p)$.

The pullback of the map $df_p : T_p\mathbb{R}^m \to T_q\mathbb{R}^n$ is

$$(df_p)^* : \Lambda^k(T_q^*\mathbb{R}^n) \to \Lambda^k(T_p^*\mathbb{R}^m). \tag{4.197}$$

Suppose you have a k-form ω on V.

$$\omega \in \Omega^k(V), \tag{4.198}$$

$$\omega_q \in \Lambda^k(T_q^*\mathbb{R}^n). \tag{4.199}$$

Then

$$(df_p)^* w_q \in \Lambda^k(T_p^*\mathbb{R}^m). \tag{4.200}$$

Definition 4.44. $f^*\omega$ is the k-from whose value at $p \in U$ is $(df_p)^*\omega_q$.

We consider two examples. Suppose $\phi \in \Omega^0(V) = \mathcal{C}^\infty(V)$. Then $f^*\phi(p) = \phi(q)$, so $f^*\phi = \phi \circ f$, where $f : U \to V$ and $\phi : V \to \mathbb{R}$.

Again, suppose that $\phi \in \Omega^0(V) = \mathcal{C}^\infty(V)$. What is $f^* d\phi$? Let $f(p) = q$. We have the map $d\phi_q : T_p\mathbb{R}^n \to T_c\mathbb{R} = \mathbb{R}$, where $c = \phi(q)$. So,

$$\begin{aligned}
(df_p)^*(d\phi)_q &= d\phi_q \circ df_p \\
&= d(\phi \circ f)_p.
\end{aligned} \tag{4.201}$$

Therefore,

$$f^* d\phi = df^* \phi. \tag{4.202}$$

Suppose that $\mu \in \Omega^k(V)$ and $\nu \in \Omega^e ll(V)$. Then

$$\begin{aligned}(f^*(\mu \wedge \nu))_p &= (df_p)^*(\mu_q \wedge \nu_q) \\ &= (df_p)^* \mu_q \wedge (df_p^*) \nu_q.\end{aligned} \tag{4.203}$$

Hence,

$$f^*(\mu \wedge \nu) = f^* \mu \wedge f^* \nu. \tag{4.204}$$

We now obtain a coordinate formula for f^*.

Take $\omega \in \Omega^k(V)$. We can write $\omega = \sum a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, where each $a_I \in \mathcal{C}^\infty(U)$. Then

$$\begin{aligned}f^* \omega &= \sum f^* a_I f^* dx_{i_1} \wedge \cdots \wedge f^* dx_{i_k} \\ &= \sum f^* a_I df_{i_1} \wedge \cdots \wedge df_{i_k},\end{aligned} \tag{4.205}$$

where we used the result that $f^* dx_i = dx_i \circ f = df_i$.

Note that $df_i = \sum \frac{\partial f_i}{\partial x_j} dx_j$, where $\frac{\partial f_i}{\partial x_j} \in \mathcal{C}^\infty(U)$. Also, $f^* a_I = a_I \circ f \in \mathcal{C}^\infty(U)$, which shows that

$$f^* \omega \in \Omega^k(U). \tag{4.206}$$

The following theorem states a very useful property of the pullback operator.

Theorem 4.45. *Let $\omega \in \Omega^k(V)$. Then,*

$$df^* \omega = f^* d\omega. \tag{4.207}$$

Proof. We have already checked this for $\omega = \phi \in \mathcal{C}^\infty(V)$, $k = 0$ already. We now prove the general case.

We can write $\omega = \sum a_I dx_I$. Then

$$f^* \omega = \sum f^* a_I df_{i_1} \wedge \cdots \wedge df_{i_k}. \tag{4.208}$$

So,

$$\begin{aligned}df^* \omega = &\sum df^* a_I \wedge df_{i_1} \wedge \cdots \wedge df_{i_k} \\ &+ \sum f^* a_I \wedge d(df_{i_1} \wedge \cdots \wedge df_{i_k})\end{aligned} \tag{4.209}$$

Note that

$$d(df_{i_1} \wedge \cdots \wedge df_{i_k}) = \sum_{r=1}^{k} (-1)^{r-1} df_{i_1} \wedge \cdots \wedge d(df_{i_r}) \wedge \cdots \wedge df_{i_k}. \tag{4.210}$$

We know that $d(df_{i_r}) = 0$, so

$$\begin{aligned}
df^*\omega &= \sum_I df^* a_I \wedge df_{i_1} \wedge \cdots \wedge df_{i_k} \\
&= \sum_I f^* da_I \wedge f^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\
&= f^*(\sum da_I \wedge dx_I) \\
&= f^* d\omega.
\end{aligned} \tag{4.211}$$

□