Lecture 22

In $\mathbb{R}^3$ we had the operators grad, div, and curl. What are the analogues in $\mathbb{R}^n$? Answering this question is the goal of today's lecture.

4.9 Tangent Spaces and k-forms

Let $p \in \mathbb{R}^n$.

Definition 4.36. The *tangent space of* p *in* $\mathbb{R}^n$ is

$$T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}. \tag{4.144}$$

We identify the tangent space with $\mathbb{R}^n$ via the identification

$$T_p\mathbb{R}^n \cong \mathbb{R}^n \tag{4.145}$$
$$(p, v) \to v. \tag{4.146}$$

Via this identification, the vector space structure on $\mathbb{R}^n$ gives a vector space structure on $T_p\mathbb{R}^n$.

Let U be an open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be a $\mathcal{C}^1$ map. Also, let $p \in U$ and define $q = f(p)$. We define a linear map

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^m \tag{4.147}$$

according to the following diagram:

$$\begin{array}{ccc} T_p\mathbb{R}^n & \xrightarrow{df_p} & T_q\mathbb{R}^m \\ \cong\big\downarrow & & \cong\big\uparrow \\ \mathbb{R}^n & \xrightarrow{Df(p)} & \mathbb{R}^m. \end{array} \tag{4.148}$$

So,

$$df_p(p, v) = (q, Df(p)v). \tag{4.149}$$

Definition 4.37. The *cotangent space of* $\mathbb{R}^n$ *at* p is the space

$$T_p^*\mathbb{R}^n \equiv (T_p\mathbb{R}^n)^*, \tag{4.150}$$

which is the dual of the tangent space of $\mathbb{R}^n$ at p.

Definition 4.38. Let U be an open subset of $\mathbb{R}^n$. A *k-form on U* is a function ω which assigns to every point $p \in U$ an element ω_p of $\Lambda^k(T_p^*\mathbb{R}^n)$ (the kth exterior power of $T_p^*\mathbb{R}^n$).

Let us look at a simple example. Let $f \in \mathcal{C}^\infty(U)$, $p \in U$, and $c = f(p)$. The map

$$df_p : T_p\mathbb{R}^n \to T_c\mathbb{R} = \mathbb{R} \tag{4.151}$$

is a linear map of $T_p\mathbb{R}^n$ into $\mathbb{R}$. That is, $df_p \in T_p^*\mathbb{R}^n$. So, df is the one-form on U which assigns to every $p \in U$ the linear map

$$df_p \in T_p^*\mathbb{R}^n = \Lambda^1(T_p^*\mathbb{R}). \tag{4.152}$$

As a second example, let $f, g \in \mathcal{C}^\infty(U)$. Then gdf is the one-form that maps

$$p \in U \to g(p)df_p \in \Lambda^1(T_p^*\mathbb{R}^n). \tag{4.153}$$

As a third example, let $f, g \in \mathcal{C}^\infty(U)$. Then $\omega = df \wedge dg$ is the two-form that maps

$$p \in U \to df_p \wedge dg_p. \tag{4.154}$$

Note that $df_p, dg_p \in T_p^*\mathbb{R}$, so $df_p \wedge dg_p \in \Lambda^2(T_p^*\mathbb{R}^n)$.

As a fourth and final example, let $f_1, \dots, f_k \in \mathcal{C}^\infty(U)$. Then $df_1 \wedge \cdots \wedge df_k$ is the k-form that maps

$$p \in U \to (df_1)_p \wedge \cdots \wedge (df_k)_p. \tag{4.155}$$

Note that each $(df_i)_p \in T_p^*\mathbb{R}^n$, so $(df_1)_p \wedge \cdots \wedge (df_k)_p \in \Lambda^k(T_p^*\mathbb{R}^n)$.

Let us now look at what k-forms look like in coordinates. Let $e_1, \dots, d_n$ be the standard basis of $\mathbb{R}^n$. Let $p \in U$ and let $v_i = (p, e_i)$ for each i. Then, the vectors $v_1, \dots, v_n$ are a basis of $T_p\mathbb{R}^n$.

Suppose we have a map $f \in \mathcal{C}^\infty(U)$. What is $df_p(v_i)$?

$$df_p(v_i) = De_i f(p) = \frac{\partial f}{\partial x_i}(p). \tag{4.156}$$

In particular, letting x_i be the ith coordinate function,

$$(dx_i)_p(v_j) = \frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.157}$$

So, $(dx_1)_p, \dots, (dx_n)_p$ is the basis of $T_p^*\mathbb{R}^n$ dual to $v_1, \dots, v_n$.

For any $f \in \mathcal{C}^\infty(U)$,

$$\begin{aligned} df_p(v_j) &= \frac{\partial f}{\partial x_j}(p) \\ &= \left(\sum_i \frac{\partial f}{\partial x_i}(p)(dx_i)_p \right)(v_j) \\ \implies df_p &= \sum \frac{\partial f}{\partial x_i}(p)(dx_i)_p \\ \implies df &= \sum \frac{\partial f}{\partial x_i} dx_i. \end{aligned} \tag{4.158}$$

Since $(dx_1)_p, \ldots, (dx_n)_p$ is a basis of $T_p^*\mathbb{R}^n$, the wedge products

$$(dx_I)_p = (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p, \;\; 1 \le i_1 < \cdots < i_k \le n, \tag{4.159}$$

(I strictly increasing) are a basis of $\Lambda^k(T_p^*\mathbb{R}^n)$.

Therefore, any element w_p of $\Lambda^k(T_p^*\mathbb{R}^n)$ can be written uniquely as a sum

$$\omega_p = \sum a_I(p)(dx_I)_p, \;\; a_I(p) \in \mathbb{R}, \tag{4.160}$$

where the I's are strictly increasing. Hence, any k-form can be written uniquely as a sum

$$\omega = \sum a_I dx_I, \;\; I \text{ strictly increasing}, \tag{4.161}$$

where each a_I is a real-valued function on U. That is, $a_I : U \to \mathbb{R}$.

Definition 4.39. The k-form ω is $\mathcal{C}^r(U)$ if each $a_I \in \mathcal{C}^r(U)$.

Just to simplify our discussion, from now on we will always take k-forms that are $\mathcal{C}^\infty$.

Definition 4.40. We define

$$\Omega^k(U) = \text{ the set of all } \mathcal{C}^\infty \text{ } k\text{-forms}. \tag{4.162}$$

So, $\omega \in \Omega^k(U)$ implies that $\omega = \sum a_I dx_I$, where $a_I \in \mathcal{C}^\infty(U)$.

Let us now study some basic operations on k-forms.

1. Let $\omega \in \Omega^k(U)$ and let $f \in \mathcal{C}^\infty(U)$. Then $f\omega \in \Omega^k(U)$ is the k-form that maps

$$p \in U \to f(p)\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n). \tag{4.163}$$

2. Let $\omega_i \in \Omega^k(U), \; i = 1, 2$. Then $\omega_1 + \omega_2$ is the k-form that maps

$$p \in U \to (\omega_1)_p + (\omega_2)_p \in \Lambda^k(T_p^*\mathbb{R}^n). \tag{4.164}$$

3. Let $\omega_i \in \Omega^{k_i}(U), \; i = 1, 2$, and $k = k_1 + k_2$. Then $w_1 \wedge \omega_2 \in \Omega^k(U)$ is the k-form that maps

$$p \in U \to (\omega_1)_p \wedge (\omega_2)_p \in \Lambda^k(T_p^*\mathbb{R}^n), \tag{4.165}$$

since $(\omega_i)_p \in \Lambda^{k_i}(T_p^*\mathbb{R}^n)$.

Definition 4.41. We find it convenient to define $\Lambda^0(T_p^*\mathbb{R}^n) = \mathbb{R}$.

A zero-form f on U is just a real-valued function, so $\Omega^0(U) = \mathcal{C}^\infty(\mathbb{R})$.

Take $f \in \mathcal{C}^\infty(U)$ and $df \in \Omega^1(U)$. This gives an operation

$$d : \Omega^0(U) \to \Omega^1(U), \tag{4.166}$$

$$f \to df. \tag{4.167}$$

Let $f, g \in \mathcal{C}^\infty(U)$ (that is, take f, g to be zero-forms). Then $d(fg) = gdf + fdg$. We can think of this operation as a slightly different notation for the gradient operation.

The maps $d : \Omega^k(U) \to \Omega^{k+1}(U), \ k = 0, \dots, (n-1)$ give n vector calculus operations.

If $\omega \in \Omega^k(U)$, then ω can be written uniquely as the sum

$$\omega = \sum a_I dx_I, \ I \text{ strictly increasing}, \tag{4.168}$$

where each $a_I \in \mathcal{C}^\infty(U)$. We define

$$d\omega = \sum da_I \wedge dx_I. \tag{4.169}$$

This operator is the unique operator with the following properties:

1. For $k = 0$, this is the operation we already defined, $df = \sum \frac{\partial f}{\partial x_i} dx_i$.
2. If $\omega \in \Omega^k$, then $d(d\omega) = 0$.
3. If $\omega_i \in \Omega^{k_i}(U), i = 1, 2$, then $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2$.

Let $a \in \mathcal{C}^\infty(U)$, and $adx_I \in \Omega^k(U)$, I strictly increasing. Then

$$d(adx_I) = da \wedge dx_I. \tag{4.170}$$

Suppose that I is not strictly increasing. Then

$$\begin{aligned} dx_I &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= 0 \ \text{ if } i_r = i_s. \end{aligned} \tag{4.171}$$

If there are no repetitions, then there exists $\sigma \in S_k$ such that $J = I^\sigma$ is strictly increasing. Then

$$dx_J = (-1)^\sigma dx_I, \tag{4.172}$$

so

$$\begin{aligned} d(adx_I) &= (-1)^\sigma d(adx_J) \\ &= (-1)^\sigma da \wedge dx_J \\ &= da \wedge dx_I. \end{aligned} \tag{4.173}$$

Putting this altogether, for *every* multi-index I of length k,

$$d(adx_I) = da \wedge dx_I. \tag{4.174}$$