Problem set 3 **due Nov. 18**

(1) In the MLA notes, §3, Exercise 7.

(2) In the MLA notes, §4, Exercise 5.

(3) Let V be a 3-dimensional vector space, $\langle v, w\rangle$ an inner product on V and $\Omega \in \Lambda^3(V^*)$, $\Omega \neq 0$.

(a) Given $\mu \in \Lambda^2(V^*)$ show that there exists a unique vector, $v_\mu \in V$, such that for all $\ell \in V^*$:

$$\mu \wedge \ell = \ell(v_\mu)\Omega\,. \qquad (**)$$

Hint: It's clear that $\mu \wedge \ell = c_\ell\,\Omega$ for some constant, c_ℓ, depending on ℓ. Show that this constant depends *linearly* on ℓ. Then show that there exists a unique vector $v_\mu \in V$ with the property:

$$c_\ell = \ell(v_\mu)$$

for all $\ell \in V^*$.

(b) For $v \in V$, let $\ell_v \in V^*$ be the linear functional

$$w \in V \to \langle v, w\rangle\,.$$

Show how to define a cross product on V by requiring that

$$v_1 \times v_2 = v_\mu \Leftrightarrow \mu = \ell_{v_1} \wedge \ell_{v_2}\,.$$

Show that this cross product is linear in v_1 and v_2 and satisfies $v_1 \times v_2 = -v_2 \times v_1$.

(c) Let $V = \mathbb{R}^3$. Show that if $\langle v, w\rangle$ is the Euclidean inner product on $\mathbb{R}^3$, e_1, e_2, and e_3, the standard basis vectors of $\mathbb{R}^3$, and $\Omega = e_1 \wedge e_2 \wedge e_3$ the standard volume form, then this cross product is the *standard* cross product.

(4) Let U be an open subset of $\mathbb{R}^3$ and let

$$\begin{aligned}\mu_1 &= dx_2 \wedge dx_3\\ \mu_2 &= dx_3 \wedge dx_1\end{aligned}$$

and

$$\mu_3 = dx_1 \wedge dx_2\,.$$

(a) If $f : U \to \mathbb{R}$ is a function of class C^1 show that $df = G_1\,dx_1 + G_2\,dx_2 + G_3\,dx_3$ where $G = (G_1, G_2, G_3) = \mathrm{grad} f$.

(b) If $\omega = F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3$ is a one-form on U of class C^1 show that $d\omega = G_1\mu_1 + G_2\mu_2 + G_3\mu_3$ where $G = \text{curl}F$.

(c) If $\omega = F_1\mu_1 + F_2\mu_2 + F_3\mu_3$ is a two-form on U of class C^1 show that $d\omega = g\, dx_1 \wedge dx_2 \wedge dx_3$ where $g = \text{div}(F)$.