Problem Set 5

1. Given a series of real numbers $\sum_{k=1}^{\infty} x_k$, define another series $\sum_l y_l$ by

$$y_1 = x_1,$$
$$y_2 = \frac{1}{2} x_2,$$
$$\cdots$$
$$y_n = \frac{1}{n(n-1)} x_2 + \frac{2}{n(n-1)} x_3 + \cdots + \frac{n-2}{n(n-1)} x_{n-1} + \frac{1}{n} x_n.$$

   Show that if $\sum_{k=1}^{\infty} x_k$ converges, then $\sum_{k=1}^{\infty} y_k$ converges to the same value.
   On the other hand, the new series may converge even though the original one doesn't. This can be used to assign values to some divergent series. Show that in this generalized sense, one could say that the following makes sense:
$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots = \frac{1}{2}.$$

   (5 points)

2. Problem 16 on page 81, part (a) only. The problem assumes the existence of square roots (which we have indeed proved previously). (3 points)

3. The proof of Theorem 3.43 in the book is not particularly simple. Find an alternative one which directly relies on the definition of convergence, and on the fact that the partial sums $s_n = c_1 + \cdots + c_n$ of an alternating series satisfy $s_1 \leq s_3 \leq s_5 \leq \cdots \leq s_6 \leq s_4 \leq s_2$. (**This problem will have to be written up carefully in LaTeX.**) (3.5 points)

4. Take the convergent series from Example 3.53. It is shown there how to rearrange it so that it converges to a different number. Find another **explicit** rearrangement so that the rearranged series doesn't converge at all. Note that following our usual terminology, going to $+\infty$ also counts as "does not converge". (4 points)

Total: $5 + 3 + 3.5 + 4 = 15.5$ points.

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