18.100B : Fall 2010 : Section R2
Homework 7

Due Tuesday, October 26, 1pm

Reading: Tue Oct.19 : continuity, Rudin 4.1-12
Thu Oct.21 : Quiz 2 (covering Rudin sections 2.45-47 and 3), ℓ^p spaces.

1 . (a) Problem 1, page 98 in Rudin
(b) Problem 3, page 98 in Rudin

2 . Let (X, d) be a metric space. Fix $x_0 \in X$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$. Show that the function $X \to \mathbb{R}$ defined by $x \mapsto g(d(x, x_0))$ is continuous.

3 . Let (S, d_S) be a set equipped with the discrete metric (i.e. $d_S(t, r) = 1$ for $t \neq r$).

(a) Show that any map $f : S \to X$ into another metric space X is continuous; using the definition of continuity by sequences.
(b) Show that any map $f : S \to X$ into another metric space X is continuous; using the definition of continuity by ϵ- and δ-balls.
(c) Which maps $f : \mathbb{R} \to S$ are continuous? (Give an easy characterization and prove it.)

4 . Consider the function $h : \mathbb{Q} \to \mathbb{R}$ given by

$$h(x) = \begin{cases} 0 & ; x^2 < 2, \\ 1 & ; x^2 > 2. \end{cases}$$

(a) Is h continuous?
(b) Can h be continuously extended to $\tilde{h} : \mathbb{R} \to \mathbb{R}$? (I.e. such that $\tilde{h}(x) = h(x)$ for all $x \in \mathbb{Q}$.)

5 . Prove that the function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ is continuous by following the steps below.

(a) Fix $T > 0$ and $\varepsilon > 0$. Show that there exists an $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} \frac{t^n}{n!} < \varepsilon \qquad \forall t \in [0, T].$$

[*Hint*: The series for e^T converges.]

(b) Show continuity at $z \in \mathbb{C}$ by splitting

$$e^z - e^x = \sum_{n=1}^{N-1} \frac{1}{n!}(z^n - x^n) + \sum_{n=N}^{\infty} \frac{1}{n!} z^n - \sum_{n=N}^{\infty} \frac{1}{n!} x^n.$$

[*Hint*: First use (a) with $T = |z| + 1$, then use the fact that polynomials are continuous.]