18.100A Fall 2012: Assignment 23

The rules are the same as for previous assignments.

Reading Mon.: 25.1-.3 Constructing closed and open sets; characterization of compact sets.

1. (2) Work 25.1/4ab, using and citing theorems in Chapter 25.

2. (2) Work 25.2/2; assume the set S is non-empty.

This is an improvement on the result in 24.7/1 (on Assignment 22). Once again, you are asked to prove there is a point $\mathbf{a} \,\epsilon\, S$ which is closest to $\mathbf{0}$, in the sense that no other point is closer.

(You have to adapt the Extremal Value Theorem (24.7B) to the non-compact set S; it's analogous to what you had to do in $\mathbf{R}^1$ for Problem 1, Exam 2. You will need the theorems of 25.1 and 25.2, rather than the definitions.)

3. (2: .5, 1.5)

a) Let P be the graph in $\mathbf{R}^2$ of the parabola $y = 2x^2 - 1$. Discuss, with proof, whether it is open, closed, compact, or none of these. Use theorems in Chapter 25; don't go back to definitions if you don't have to.

b) Consider the sequence $\mathbf{x}_n = (\cos n, \cos 2n), \quad n = 0, 1, 2, \ldots$ in $\mathbf{R}^2$.

Using the theorems in Chapter 25, and part (a), prove it has a subsequence which converges to a point $\mathbf{a}$ on the parabola in part (a).

(This is like Problem 5 in Ass't. 21, except you now have more powerful theorems in Chapter 25 that you can use directly or adapt to this problem.)

4. (2) We can think of a function $w = f(\mathbf{x})$ defined for all $\mathbf{x} \,\epsilon\, \mathbf{R}^2$ as giving a map $f : \mathbf{R}^2 \to \mathbf{R}^1$. If $S \subset \mathbf{R}^1$, we define

$$f^{-1}(S) = \{\mathbf{x} \,\epsilon\, \mathbf{R}^2 : f(\mathbf{x}) \,\epsilon\, S\} \; .$$

Assume $f(\mathbf{x})$ is continuous; prove that if S is closed in $\mathbf{R}^1$, then $f^{-1}(S)$ is closed in $\mathbf{R}^2$.

(This is a general property of continuous maps; conversely, if the statement is true for all closed subsets S of R^1, then it follows that the function $f(\mathbf{x})$ is continuous.)

Focus on what you have to prove about $f^{-1}(S)$; use (1c) in Def'n. 25.1A as the definition of cluster point. Some care is needed.

see over for reading and problems for Wed. class →

Reading Wed.: 26.1-.2 to top third of p. 379

Functions defined by integrals with a parameter: continuity, differentiation under the integral sign.

5. (1.5: 1, .5) For the function $\phi(x) = \displaystyle\int_0^1 \frac{e^{x+t}}{1+xt}\, dt$

(a) indicating the reasoning, find the largest x-interval on which the Continuity Theorem 26.1 predicts the integral will be continuous; your answer should take account of the t-values being restricted to a certain interval.

(b) Using the information provided by part (a), find $\displaystyle\lim_{x\to 0} \int_0^1 \frac{e^{x+t}}{1+xt}\, dt$. Cite the theorem(s) I hope you are using.

6. (2.5: 1, .5, 1) Let $\phi(x) = \displaystyle\int_0^\pi \sin(xt)\, dt$.

(a) The Derivative Theorem 26.2 says $\phi'(x)$ exists for all x and gives a formula for it. *Using the formula*, calculate $\phi'(x)$ explicitly for all x, including $x = 0$, using standard integration techniques; your final answer should not have any integral signs in it, and should include all values of x.

(b) Verify your calculation of $\phi'(x)$ for $x \neq 0$ by first calculating $\phi(x)$, $x \neq 0$ explicitly, and then differentiating it by the usual rules.

(c) The Continuity Theorem predicts $\phi'(x)$ will be continuous for all x. The explicit formula for it shows it is continuous if $x \neq 0$; show it is continuous also at 0.

Do this, preferably by using the standard quadratic approximations to $\sin u$ and $\cos u$ at $u = 0$ given by their Taylor series; or else by L'Hospital's rule.

7. (3) Work P26-1, showing that the Bessel function $J_0(x)$ of order 0, given in the form of a definite integral involving a parameter x, is the solution to Bessel's ODE, with the given initial conditions.

You will need to differentiate $J_0(x)$ twice; verify hypotheses each time, and verify the initial conditions are satisfied.

After substituting into Bessel's ODE and combining the integrals you get a very big definite integral, which is supposed to have the value 0, if the function $J_0(x)$ really does solve the ODE.

Stop at this point and check your work carefully – the definite integral must be calculated correctly. Then try showing its value is 0 by using one of the standard techniques for transforming or evaluating integrals in Chapter 20. More than one technique will work.

If nothing strikes you after a few minutes, try sleeping on it and looking at the integral again in the morning.