18.100A Fall 2012: Assignment 17

Directions: If you collaborate, list collaborators and write up the solutions independently. Consulting solutions to problem sets of previous years is not allowed. Cite significant theorems or examples when needed for reasoning.

Reading: 20.6, 21.1-.2 Growth rate; improper integrals, convergence theorems.

1. (2) The *Riemann Hypothesis* (dating from 1859) is the Holy Grail of unsolved problems in number theory; it's one of the 7 Millenium Problems in mathematics – $1M reward for solving any of them. Going fast, only six are left, better get started.

Let $\pi(x) = $ the number of primes $\leq x$, so $\pi(10) = 4$ for instance (note that 1 is not considered a prime).

The classic problem in number theory is to find a simple, smooth function $f(x)$ asymptotic to $\pi(x)$ as $x \to \infty$, and to estimate how well it approximates $\pi(x)$. Euler and Gauss guessed functions, but it took 150 years to prove that both were right:

The Prime Number Theorem.

$$\text{Euler:} \quad \pi(x) \ \sim \ \frac{x}{\ln x} \ ; \qquad \text{Gauss:} \quad \pi(x) \ \sim \ \text{Li}(x), \quad \text{where } \text{Li}(x) = \int_2^x \frac{dt}{\ln t} \ .$$

(a) Prove the Euler and Gauss functions are asymptotic to each other, so either form of the Prime Number Theorem implies the other.

Though the two forms are equivalent, Gauss' function $\text{Li}(x)$ is known to give a much more accurate estimate of $\pi(x)$ than Euler's. For example, if we write

$$(1) \qquad\qquad \pi(x) = \text{Li}(x) + e(x) \qquad (e(x) = \text{ the "error")}$$

then it turns out that $e(x)$ keeps changing sign as you let x increase.

Riemann conjectured in 1859 that there was a constant C such that for all $x > 0$,

$$(2) \qquad\qquad |e(x)| < C\sqrt{x} \ln x \ , \qquad \textbf{The Riemann Hypothesis}.$$

In spite of strenuous efforts, no one has proved this. The best result in this direction was proved by Norman Levinson (MIT Math Dept) around 1965.

Another form of the Riemann hypothesis says that $e(x)$ is negligible compared with any power of x bigger than $1/2$:

$$(3) \qquad\qquad \text{Given any } \epsilon > 0, \quad \lim_{x \to \infty} \frac{|e(x)|}{x^{(1/2)+\epsilon}} \ = \ 0.$$

b) Which form of the Hypothesis, (2) or (3), is the stronger statement? Prove your answer.

Then you can try proving either form for extra credit (and 1000K).

The remaining problems are mostly about improper integrals. Before doing the first two, study the Examples in the text. If you are a bit rusty on improper integrals, try a few of the Questions first.

2. (2) Work 21.1/2; do it two different ways:

a) Change the variable to convert it to an integral of the first kind, then use comparison.

b) Do the integration, then evaluate the limit you get, either by changing variable, or converting it to a standard indeterminate form.

3. (3: 1.5,1.5) In both of these, k is any real number.

a) Work 21.2/1c, using asymptotic comparison at both ends.

b) Work 21.2/1e.

4. (1) Work 21.2/4, using the following comments:

The statement is false, so you should look for a counterexample. Find one which is non-negative and continuous.

You can either give the function explicitly, or make a clearly drawn graph of it with enough accompanying details given so that a computer could draw the graph.

(If stuck, begin by finding a counterexample having discontinuities. Note that altering the value of a function at a single isolated point does not change its integrability, nor the value of its integral (as in Ass't 15, Problem 4: 18.3/1).

5. (2) Work: Problem 20-1abcd. Use the following hints and changes:

a) Use a change of variable.

b) Use theorems, verify hypothees.

d) Do it just for $x \geq 0$. Omit the second sentence, but tell how you would define the number π in terms of $T(x)$, and prove it exists. (For this, you can use one of the theorems stated in the reading from Chapter 21.)

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