5.3   The Wave Equation and Staggered Leapfrog

This section focuses on the **second-order wave equation $u_{tt} = c^2 u_{xx}$**. We find the exact solution $u(x, t)$. Accuracy and stability are confirmed for the leapfrog method (centered second differences in t and x). This two-step method requires that we rethink the growth factor G, which was clear for a single step. The result will be $p = 2$ for the order of accuracy, and $c\Delta t/\Delta x \le 1$ for stability.

It is useful to rewrite the wave equation as a *first-order system*. The components v_1 and v_2 of the vector unknown can be $\partial u/\partial t$ and $c\,\partial u/\partial x$. Then we are back to a single-step growth factor, but G is now a 2 by 2 matrix.

Second-order accuracy extends to this system $v_t = Av_x$ if we use a **staggered mesh**. The mesh for v_2 lies in between the mesh for v_1. This has become the standard method in computational electromagnetics (solving Maxwell's equations). The physical laws relating the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ are beautifully copied by the difference equations on a staggered mesh. The mesh becomes especially important in more space dimensions (x-y and x-y-z), and in finite volume methods.

This section goes beyond the one-way wave equation in at least five ways:

1.   *Two characteristic lines $x + ct = C_{\text{left}}$ and $x - ct = C_{\text{right}}$ go through each (x, t).*

2.   The *leapfrog method* involves three time levels $t + \Delta t$, t, and $t - \Delta t$.

3.   *First-order systems* have vector unknowns $v(x, t)$ and growth matrices G.

4.   *Staggered grids* give the much-used FDTD method for Maxwell's equations.

5.   *More space dimensions* lead to new CFL and vN stability conditions on Δt.

With $-\infty < x < \infty$, we don't yet have boundary conditions in space. And we are not facing real problems like $u_{tt} = c^2(x)\, u_{xx} + Fe^{ikx}$, with a high frequency forcing terms ($k \gg 1$) and a varying speed $c(x)$.

Solution of the Wave Equation

Exactly as for the one-way equation $u_t = cu_x$, we solve the two-way wave equation $u_{tt} = c^2 u_{xx}$ for each pure exponential. That allows us to **separate the variables**. The space variable is in e^{ikx}, and we look for solutions $\boldsymbol{u(x,t) = G(t)e^{ikx}}$:

$$\textbf{Each } \boldsymbol{k} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{becomes} \quad \frac{d^2 G}{dt^2} e^{ikx} = i^2 c^2 k^2 G\, e^{ikx}\,. \tag{1}$$

Thus $G_{tt} = i^2 c^2 k^2 G$. This second-order equation has two solutions, $G_{\text{left}} = e^{ickt}$ and $G_{\text{right}} = e^{-ickt}$. So there are two waves with speed c:

$$\textbf{Pure waves} \quad \boldsymbol{u_{\text{left}}(x,t) = e^{ik(x+ct)}} \quad \text{and} \quad \boldsymbol{u_{\text{right}}(x,t) = e^{ik(x-ct)}}\,. \tag{2}$$

Combinations of left-going waves $e^{ik(x+ct)}$ will give a general function $F_1(x+ct)$. Combinations of $e^{ik(x-ct)}$ give $F_2(x-ct)$. The complete solution includes both:

$$u(x,t) = u_{\text{left}}(x,t) + u_{\text{right}}(x,t) = F_1(x+ct) + F_2(x-ct)\,. \tag{3}$$

We need those two functions to match an initial shape $u(x,0)$ and velocity $u_t(x,0)$:

$$\textbf{At } t = 0 \quad u(x,0) = F_1(x) + F_2(x) \quad \text{and} \quad u_t(x,0) = c\,F_1'(x) - c\,F_2'(x)\,. \tag{4}$$

Solving for F_1 and F_2 gives the unique solution that matches $u(x,0)$ and $u_t(x,0)$:

$$\textbf{Solution} \quad u(x,t) = \frac{u(x+ct,0) + u(x-ct,0)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(x,0)\,dx\,. \tag{5}$$

The "domain of dependence" of $u(x,t)$ includes the initial values from $x-ct$ to $x+ct$. That domain is on the left side of Figure 5.9, bounded by the characteristic lines.

Example 1   Starting with zero velocity, $u_t(x,0) = 0$, the integrated term in formula (5) is zero. A step function $S(x)$ (*wall of water*) will travel left and right along characteristic lines, as in Figure 5.8. It reaches the points $x = 1$ and $x = -1$ at time $t = 1/c$:

$$\textbf{Two walls} \quad u(x,t) = \frac{1}{2}S(x+ct) + \frac{1}{2}S(x-ct) = \left\{0 \text{ or } \frac{1}{2} \text{ or } 1\right\}.$$

By time t, the initial jump at $x = 0$ affects the solution between $x = -ct$ and $x = ct$. That is the "domain of influence" of the point $x = 0, t = 0$.
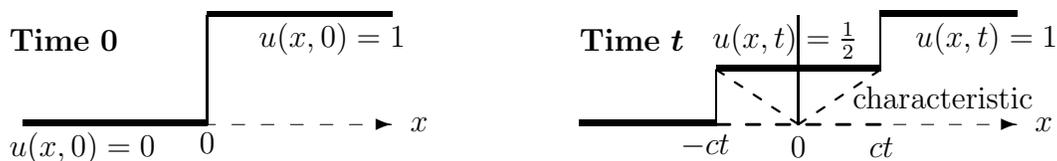
Figure 5.8: Two-wall solution to the wave equation starting from a step function.

The Semidiscrete Wave Equation

Let me start by discretizing only the space derivative u_{xx}. The second difference $U_{j+1} - 2U_j + U_{j-1}$ is the natural choice, divided by $(\Delta x)^2$. For the approximations $U_j(t)$ at the meshpoints $x = j\Delta x$, we have a family of ODEs in the time direction (*method of lines*):

$$\textbf{Semidiscrete } u_{tt} = c^2 u_{xx} \qquad U_j'' = \frac{c^2}{(\Delta x)^2}(U_{j+1} - 2U_j + U_{j-1})\,. \tag{6}$$

Again we follow every exponential, looking for $U_j(t) = G(t)e^{ikj\Delta x}$. Substitute into (6) and cancel the common factor $e^{ikj\Delta x}$. Instead of $G_{tt} = -c^2 k^2 G$ we have

Growth equation $G_{tt} = \dfrac{c^2}{(\Delta x)^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x})G = -\dfrac{c^2}{(\Delta x)^2}(2 - 2\cos k\Delta x)(G).$

The correct right side $-c^2 k^2 G$ is multiplied by a factor F^2. This F^2 turns up so often that we need to recognize it! Use $2 - 2\cos\theta = 4\sin^2(\theta/2)$:

Sinc squared $\quad F^2 = \dfrac{2 - 2\cos k\Delta x}{k^2(\Delta x)^2} = \dfrac{4\sin^2(k\Delta x/2)}{k^2(\Delta x)^2} = \left(\dfrac{\sin(k\Delta x/2)}{k\Delta x/2}\right)^2.$ $\qquad$ (8)

The "*sinc function*" is defined as $\sin\theta$ divided by θ. When $\theta = k\Delta x$ is small, this is $1 + 0(\theta^2)$. Then F^2 near 1 and equation (7) is close to the correct $G_{tt} = -c^2 k^2 G$.

For every $k\Delta x$, the growth equation (7) has two exponential solutions:

Semidiscrete growth $\qquad G_{tt} = -c^2 F^2 k^2 G \quad$ gives $\quad G(t) = e^{\pm icFkt}.$ $\qquad$ (9)

The wave speed c is multiplied by F to give the numerical "phase velocity" cF. *Notice that F depends on k.* Different frequencies e^{ikx} are traveling at different speeds $cF(k)$. This is **dispersion** and we will see it again in Section 5._____.

I will mention that the "group velocity"—the derivative of cFk with respect to k—is a more important quantity than the phase velocity cF.

The semidiscrete form suggests a good algorithm for the wave equation, if we have boundary conditions (say $u = 0$ along the lines $t = 0$ and $t = \pi$). If $h = \Delta x = \frac{\pi}{n+1}$, this interval has interior meshpoints. The n by n second difference matrix is the special K from earlier chapters (but now we have $-K$):

Semidiscrete with boundaries $\qquad U''(t) = \dfrac{c^2}{(\Delta x)^2} KU.$ $\qquad$ (10)

This is just the equation $MU'' + KU = 0$ of oscillating springs in Section 2.2.

The n eigenvalues of K are positive numbers $2 - 2\cos j\Delta x$. The only change from the equation on an infinite line is that j takes only the values $1, 2, \ldots, n$. The oscillations go on forever as in (8), the energy is conserved, and now the waves bounce back from the boundaries instead of continuing out to $x = \pm\infty$.

Leapfrog from Centered Differences

A fully discrete method also approximates u_{tt} by a centered differences. This time difference "*leaps over*" the space difference at $t = n\Delta t$:

Leapfrog method $\qquad \dfrac{U_{j,n+1} - 2U_{j,n} + U_{j,n-1}}{(\Delta t)^2} = c^2 \dfrac{U_{j+1,n} - 2U_{j,n} + U_{j-1,n}}{(\Delta x)^2}.$ $\qquad$ (11)

This has two key differences from the 5-point molecule for $u_{xx} + u_{yy} = 0$ (Laplace). First, $u_{tt} - c^2 u_{xx}$ has a minus sign. Second, we have two conditions at $t = 0$ and no

conditions at a later time. We are marching forward in time (marching with Laplace's equation would be totally unstable). A separate calculation for the first time step computes $U_{j,1}$ from the initial shape $u(x, 0)$ and the velocity $u_t(x, 0)$.

The accuracy of leapfrog is normally second-order. Substitute the true $u(x, t)$ into (11), and use the Taylor series for second differences (Section 1.2). The first terms in the local error give consistency.

Second-order $\qquad u_{tt} + \dfrac{1}{12}(\Delta t)^2 u_{tttt} + \cdots = c^2 (u_{xx} + \dfrac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots). \qquad (12)$

In this case $u_{tttt} = c^2 u_{xxtt} = c^2 u_{ttxx} = c^4 u_{xxxx}$. The two sides of (12) differ by

Local discretization error $\qquad \dfrac{1}{12}[(\Delta t)^2 c^4 - (\Delta x)^2 c^2] u_{xxxx} + \cdots \qquad (13)$

Again $c\Delta t = \Delta x$ is the golden time step that follows the characteristic exactly. The two triangles in Figure 5.9 become exactly the same in this borderline case $r = 1$. The CFL reasoning shows instability for $r > 1$. We now show that $r \leq 1$ is stable.
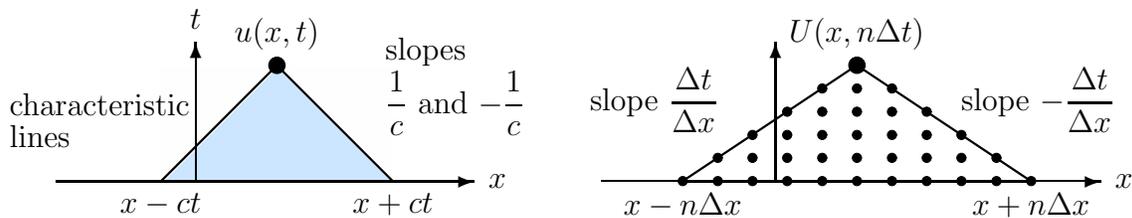
Figure 5.9: Domains of dependence: u from wave equation and U from leapfrog.

Stability of the Leapfrog Method

A difference equation must use the initial conditions in this whole interval, to have a chance of converging to $u(x, t)$. The domain of dependence for U must include the domain of dependence for u. The slopes must have $\Delta t/\Delta x \leq 1/c$. Since convergence requires stability, we have a Courant-Friedrichs-Lewy condition on Δt:

CFL condition The leapfrog method will require $\boxed{r = c\,\Delta t/\Delta x \leq 1.}$

For a double-step difference equation, we still look for pure solutions $U(x, n\Delta t) = G^n e^{ikx}$, separating time from space. In the leapfrog equation (11) this gives

$$\left[\frac{G^{n+1} - 2G^n + G^{n-1}}{(\Delta t)^2}\right] e^{ikx} = c^2 G^n \left[\frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{(\Delta x)^2}\right] e^{ikx}.$$

Set $r = c\Delta t/\Delta x$ and cancel $G^{n-1} e^{ikx}$. This leaves a quadratic equation for G:

$$G^2 - 2G + 1 = r^2 G (2\cos k\Delta x - 2). \qquad (14)$$

The two-step leapfrog equation allows two G's (of course!). For stability, both must satisfy $|G| \leq 1$ for all frequencies k. Rewrite equation (14) for G:

Growth factor equation $G^2 - 2\left[1 - r^2 + r^2 \cos k\Delta x\right] G + 1 = 0.$ (15)

The roots of $G^2 - 2aG + 1 = 0$ are $G = a \pm \sqrt{a^2 - 1}$. Everything depends on that square root giving an imaginary number, when $a^2 = [1 - r^2 + r^2 \cos k\Delta x]^2 \leq 1$:

If $a^2 \leq 1$ then $G = a \pm i\sqrt{1 - a^2}$ has $|G|^2 = a^2 + (1 - a^2) = 1.$

The CFL condition $r \leq 1$ does produce $a^2 \leq 1$ and the leapfrog method is stable:

Stability   If $r \leq 1$ then $|a| = |1 - r^2 + r^2 \cos k\Delta x| \leq (1 - r^2) + r^2 = 1.$ (16)

An unstable $r > 1$ would produce $|a| = |1 - 2r^2| > 1$ at the dangerous $k\Delta x = \pi$. Then both G's are real, and their product is 1, and one of them has $|G| > 1$.

Note 1   Suppose r is exactly 1, so that $\mathbf{c\Delta t = \Delta x}$. At this "golden ratio" we expect perfect accuracy. The middle terms $-2U_{j,n}$ and $-2r^2 U_{j,n}$ cancel in the leapfrog equation (11), leaving a complete leap over the center points at (j, n) when $r = 1$:

Exact leapfrog $U_{j,n+1} + U_{j,n-1} = U_{j+1,n} + U_{j-1,n}.$ (17)

The difference equation is satisfied by $u(x,t)$, because it is satisfied by every wave $U(x + ct)$ and $U(x - ct)$. Take $U_{j,n} = U(j\Delta x + cn\Delta t)$ and use $c\Delta t = \Delta x$:

$U_{j,n+1}$ and $U_{j+1,n}$ are both equal to $U(j\Delta x + cn\Delta t + \Delta x)$

$U_{j,n-1}$ and $U_{j-1,n}$ are both equal to $U(j\Delta x + cn\Delta t - \Delta x)$

So (17) is satisfied by all traveling waves $U(x + ct)$, and similarly by $U(x - ct)$.

Note 2   You could also apply leapfrog to the *one-way* equation $u_t = c\,u_x$:

One-way leapfrog $U_{j,n+1} - U_{j,n-1} = \dfrac{c\Delta t}{\Delta x}(U_{j+1,n} - U_{j-1,n}).$ (18)

Now the growth factor equation is $G^2 - 2(ir \sin k\Delta x)G - 1 = 0$. Problem ___ confirms that the stability condition is again $r \leq 1$. In that stable case, one growth factor is sensible and the other is strange:

$$G_1 = e^{ir \sin k\Delta x} \approx e^{ick\Delta t} \quad \text{and} \quad G_2 = -e^{-ir \sin k\Delta x} \approx -1.$$ (19)

G_1 and G_2 are *exactly on the unit circle*. With $|G| = 1$ there is no room to move. Numerical diffusion $\alpha(U_{j+1,n} - 2U_{j,n} + U_{j-1,n})$ usually adds extra stability, but not here. So leapfrog for first-order equations can be dangerous.

Section 5.4 will study the convection-diffusion equation $u_t = c\,u_x + d\,u_{xx}$.

Wave Equation in Higher Dimensions

The wave equation extends to three-dimensional space (with speed set at $c = 1$):

3D Wave equation $\qquad\qquad\qquad u_{tt} = u_{xx} + u_{yy} + u_{zz}\,.$ $\qquad\qquad$ (20)

Waves go in all directions, and the solution is a superposition of pure harmonics. These plane waves now have three wave numbers k, ℓ, m, and frequency w:

Exponential solutions $\qquad u(x, y, z, t) = e^{i(\omega t + kx + \ell y + mz)}\,.$

Substituting into the wave equation gives $\omega^2 = k^2 + \ell^2 + m^2$. So there are two frequencies $\pm \omega$. These exponential solutions combine to match the initial wave height $u(x, y, z, 0)$ and its velocity $u_t(x, y, z, 0)$.

Suppose the initial velocity is a ***three-dimensional delta function*** $\boldsymbol{\delta(x, y, z)}$:

$$\delta(x, y, z) = \delta(x)\delta(y)\delta(z) \quad \text{gives} \quad \iiint f(x, y, z)\, \delta(x, y, z)\, dV = f(0, 0, 0)\,. \qquad (21)$$

The resulting $u(x, y, z, t)$ will be the *fundamental solution* of the wave equation. It is the response to the delta function, which gives equal weight to all harmonics. Rather than computing that superposition we find it from the wave equation itself. Spherical symmetry greatly simplifies $u_{xx} + u_{yy} + u_{zz}$, when u depends only on r and t:

Symmetry produces $u(r, t)$ $\qquad\qquad \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r}\,.$ $\qquad\qquad$ (22)

Multiplying by r, this is a one-dimensional equation $(ru)_{tt} = (ru)_{rr}$! Its solutions ru will be functions of $r - t$ and $r + t$. Starting from a delta function is like sound going out from a bell, or light from a point source. *The solution is nonzero only on the sphere $r = t$.* So every point hears the bell only once, as the sound wave passes by. An impulse in 3D produces a sharp response (this is Huygen's principle).

In 2D, the solution does *not* return to zero for $t > r$. We couldn't hear or see clearly in Flatland. You might imagine a point source in two dimensions as a *line source* in the z-direction in three dimensions. The solution is independent of z, so it satisfies $u_{tt} = u_{xx} + u_{yy}$. But in three dimensions, spheres starting from sources along the line continue to hit the listener. They come from further and further away, so the solution decays—but it is not zero. The wave front passes, but waves keep coming.

EXERCISE ON EQ.(26)

Leapfrog Method in Higher Dimensions

In one dimension, two characteristics go out from each point $(x, 0)$. In 2D and 3D, a *characteristic cone* goes out from $(x, y, 0)$ and $(x, y, z, 0)$. It is essential to see how the stability condition changes from $r = c\Delta t/\Delta x \leq 1$.

The leapfrog method replaces u_{xx} and u_{yy} by centered differences at time $n\Delta t$:

Leapfrog for $u_{tt} = u_{xx} + u_{yy}$
$$\frac{U_{n+1} - 2U_n + U_{n-1}}{(\Delta t)^2} = \frac{\Delta_x^2 U_n}{(\Delta x)^2} + \frac{\Delta_y^2 U_n}{(\Delta y)^2}.$$

U_0 and U_1 come from the given initial conditions $u(x, y, 0)$ and $u_t(x, y, 0)$. We look for a solution $U_n = G^n e^{ikx} e^{i\ell y}$ with **separation of variables**. Substituting into the leapfrog equation and canceling $G^{n-1} e^{ikx} e^{i\ell y}$ produces the 2D equation for two G's:

Growth factor
$$\frac{G^2 - 2G + 1}{(\Delta t)^2} = G \frac{(2\cos k\,\Delta x - 2)}{(\Delta x)^2} + G \frac{(2\cos \ell\,\Delta y - 2)}{(\Delta y)^2}. \quad (23)$$

Again this has the form $G^2 - 2aG + 1 = 0$. You can see a in brackets:

$$G^2 - 2\left[1 - \left(\frac{\Delta t}{\Delta x}\right)^2 (1 - \cos k\,\Delta x) - \left(\frac{\Delta t}{\Delta y}\right)^2 (1 - \cos \ell\,\Delta y)\right] G + 1 = 0. \quad (24)$$

Both roots must have $|G| = 1$ for stability. This still requires $-1 \le a \le 1$. When the cosines are -1 (the dangerous value) we find the stability condition for leapfrog:

Stability
$$-1 \le 1 - 2\left(\frac{\Delta t}{\Delta x}\right)^2 - 2\left(\frac{\Delta t}{\Delta y}\right)^2 \text{ needs } \left(\frac{\Delta t}{\Delta x}\right)^2 + \left(\frac{\Delta t}{\Delta y}\right)^2 \le 1. \quad (25)$$

For $\Delta x = \Delta y$ on a square grid, this is $\Delta t \le \Delta x / \sqrt{2}$. In three dimensions it would be $\Delta t \le \Delta x / \sqrt{3}$. Those also come from the CFL condition, that the characteristic cone must lie inside the pyramid that gives the leapfrog domain of dependence. Figure 5.10 shows the cone and pyramid just touching, when $\Delta t = \Delta x / \sqrt{2}$.

Cone has circular base for $u_{tt} = u_{xx} + u_{yy}$
Pyramid has diamond base for leapfrog
Cone and pyramid go up to $(0, 0, \Delta t)$
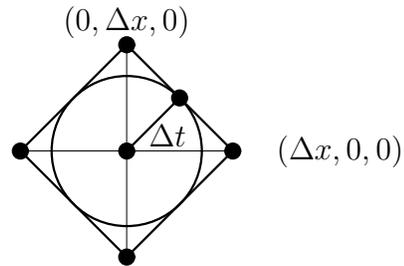
Figure 5.10: The pyramid contains and touches the cone when $(\Delta t)^2 = (\Delta x)^2/2$.

An Equivalent First-order System

I can display a system of two equations $v_t = Av_x$ that is equivalent to $u_{tt} = c^2 u_{xx}$:

First-order system
$$\frac{\partial}{\partial t} \begin{bmatrix} u_t \\ cu_x \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u_t \\ cu_x \end{bmatrix}. \quad (26)$$

The first equation recovers $u_{tt} = c^2 u_{xx}$. The second is the identity $cu_{xt} = cu_{tx}$. Notice that the 2 by 2 matrix is symmetric and its eigenvalues are the wave velocities $\pm c$.

This "symmetric hyperbolic" form $v_t = A v_x$ is useful in theory and practice. The energy $E(t) = \int \frac{1}{2} \|v(x,t)\|^2 \, dx$ is automatically constant in time! Here is the proof for any equation $v_t = A v_x$ with a symmetric matrix A:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \|v\|^2 \right) = \sum v_i \frac{\partial v_i}{\partial t} = v^{\mathrm{T}} v_t = v^{\mathrm{T}} A v_x = \frac{\partial}{\partial x} \left(\frac{1}{2} v^{\mathrm{T}} A v \right). \qquad (27)$$

When you integrate over all x, the left side is $\partial E / \partial t$. The right side is $\frac{1}{2} v^{\mathrm{T}} A v$ at the limits $x = \pm\infty$. Those limits give zero (no signal has reached that far). So the derivative of $E(t)$ is zero, and $E(t)$ stays constant.

The Euler equations of compressible flow are also a first-order system, but not linear. In physics and engineering, a linear equation deals with a small disturbance. Something from outside acts to change the equilibrium, but not by much:

    in acoustics it is a slowly moving body

    in aerodynamics it is a slender wing

    in elasticity it is a small load

    in electromagnetism it is a small source.

Below some level, the cause-effect relation is very close to linear. In acoustics, the sound speed is steady when pressure is nearly constant. In elasticity, Hooke's law holds until the geometry changes or the material begins to break down. In electromagnetism, nonlinearity comes with relativistic and quantum effects.

The case to understand has $A = $ constant matrix, with n real eigenvalues λ and eigenvectors w. The vector equation $v_t = A v_x$ will split into n scalar one-way wave equations $U_t = \lambda U_x$. When $Aw = \lambda w$ we look for $v(x,t) = U(x,t)w$:

$$v_t = A v_x \quad \text{becomes} \quad \frac{\partial U}{\partial t} w = A \frac{\partial U}{\partial x} w = \lambda \frac{\partial U}{\partial x} w \quad \text{so} \quad \frac{\partial U}{\partial t} = \lambda \frac{\partial U}{\partial x}. \qquad (28)$$

The complete solution vector is $v(x,t) = U_1(x + \lambda_1 t) w_1 + \cdots + U_n(x + \lambda_n t) w_n$. **The problem $v_t = A v_x$ has n signal speeds λ_i and it sends out n waves.**

There are n characteristic lines $x + \lambda_i t = $ constant. The wave equation has $\begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}$ and two eigenvalues $\lambda = c$ and $\lambda = -c$. The eigenvectors w are $(\frac{1}{2}, -\frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. Then the two scalar equations $U_t = \lambda U_x$ produce left and right waves:

$$\boldsymbol{\lambda_1 = \quad c} \qquad \frac{\partial}{\partial t}(u_t + c\, u_x) = \quad c \frac{\partial}{\partial x}(u_t + c\, u_x)$$

$$\boldsymbol{\lambda_2 = -c} \qquad \frac{\partial}{\partial t}(u_t - c\, u_x) = -c \frac{\partial}{\partial x}(u_t - c\, u_x). \qquad (29)$$

Each equation agrees with $u_{tt} = c^2 u_{xx}$. The one-way waves are $U(x,t) = U(x + \lambda t, 0)$. The vector solution $v(x,t)$ is recovered from $U_1 w_1 + U_2 w_2$:

$$v = \begin{bmatrix} u_t \\ c\, u_x \end{bmatrix} = (u_t + c\, u_x) \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + (u_t - c\, u_x) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}. \qquad (30)$$

A stable difference method for $v_t = A v_x$ comes from a stable method for $u_t = \pm c u_x$. Just replace c by A in Lax-Friedrichs and Lax-Wendroff, or go to leapfrog.

Leapfrog on a Staggered Grid

The discrete case should copy the continuous case. The two-step leapfrog difference equation should reduce to a pair of one-step equations. But if we don't keep the individual equations centered, they will lose second-order accuracy. **The way to center both first-order equations is to use a staggered grid** (Figure 5.11).

Please allow me to name the two components $v_1 = \boldsymbol{E}$ and $v_2 = \boldsymbol{H}$. Then the staggered grid for the wave equation matches *Yee's method for Maxwell's equations.* Yee's idea transformed the whole subject of computational electromagnetics (it is now called the FDTD method: finite differences in the time domain). Previously the moment method, which is Galerkin's method, had been dominant—but staggered grids are so natural for $\boldsymbol{E}$ and $\boldsymbol{H}$. We stay with the wave equation here, copying (26):

Maxwell in 1D $\partial E / \partial t = c\, \partial H / \partial x$ becomes $\Delta_t E / \Delta t = c \Delta_x H / \Delta x$
(normalized) $\partial H / \partial t = c\, \partial E / \partial x$ $\Delta_t H / \Delta t = c \Delta_x E / \Delta x$.   (31)

Those first derivatives of E and H are replaced by first differences. I will put E on the standard grid and H on the staggered (half-integer) grid. *Notice how all the differences are centered in Figure 5.11a. This gives second-order accuracy.*

The identities $E_{tx} = E_{xt}$ and $H_{tx} = H_{xt}$ lead to wave equations for E *and* H:

$$\begin{array}{rcl} E_t & = & cH_x \\ H_t & = & cE_x \end{array} \quad \text{becomes} \quad \begin{array}{rclclcl} E_{tt} & = & cH_{xt} & = & cH_{tx} & = & c^2 E_{xx} \\ H_{tt} & = & cE_{xt} & = & cE_{tx} & = & c^2 H_{xx} \end{array} \qquad (32)$$

In the discrete case, the identity is $\Delta_x(\Delta_t) = \Delta_t(\Delta_x)$. Differences copy derivatives.

When we eliminate H, we get the two-step leapfrog equation for E. And eliminating E gives the leapfrog equation for H. This all comes from the finite difference analogue of the cross-derivative identity $u_{tx} = u_{xt}$:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) \quad \textit{corresponds to} \quad \frac{\Delta_x(\Delta_t U)}{(\Delta x)(\Delta t)} = \frac{\Delta_t(\Delta_x U)}{(\Delta t)(\Delta x)}. \qquad (33)$$

With equal denominators, we only need to check the numerators. On any grid, the same 1's and -1's appear both ways in $\Delta x \Delta t$ and $\Delta t \Delta x$!

$$\begin{array}{ll} \Delta_x(\Delta_t U) & = & (U_{n+1,j+1} - U_{n,j+1}) & - & (U_{n+1,j} - U_{n,j}) \\ \Delta_t(\Delta_x U) & = & (U_{n+1,j+1} - U_{n+1,j}) & - & (U_{n,j+1} - U_{n,j}) \end{array}$$

You could compare ( ) with the Cauchy-Riemann equations $u_x = s_y$ and $u_y = -s_x$ for the potential $u(x, y)$ and stream function $s(x, y)$. (Those solve Laplace's equation and not the wave equation.) It would be natural to discretize the Cauchy-Riemann equations on a staggered grid.

May I emphasize that these grids are useful for many other equations too. We will see the "half-point" grid values in Section 5.____ for the flux F in the conservation law $u_t + F(u)_x = 0$, which is a nonlinear extension of the one-way wave equation. Half-point values are centrally important throughout the ***finite volume method***. Maxwell's equations in **integral form** lead to the finite integration technique [ ].

Maxwell's Equations

For electrodynamics, the number c in the wave equation is the *speed of light*. It is the same large number that appears in Einstein's $e = mc^2$. The CFL stability condition $c^2(\Delta t)^2 \leq (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ for the leapfrog method might require very small time steps (on the scale of ordinary life). But we all know that the wavelength for light is nothing like a meter or a centimeter. The leapfrog method is entirely appropriate, and we write Maxwell's equations without source terms:

Maxwell's equations in free space $\quad \dfrac{\partial E}{\partial t} = \dfrac{1}{\epsilon}\, \text{curl}\, H \quad \text{and} \quad \dfrac{\partial H}{\partial t} = -\dfrac{1}{\mu}\, \text{curl}\, E$ (34)

An important application is the reflection of a radar signal by an airplane. The region of interest is **exterior** to the plane. In principle that region extends infinitely far in all directions. In practice we compute inside a large box, and choose boundary conditions that don't reflect waves back into the box from its artificial boundary (which is a computational region and not physical).

Those **absorbing boundary conditions** [ ] are crucial to a good discretization. Chapter 7 of [ ] describes how a *"perfectly matched layer"* can select coefficients so that waves go through the boundary with very little reflection. Applications of Maxwell's equations range all the way from the Earth's electromagnetic environment to cell phones (safety of the user) to micron-scale lasers and photonics.

The first of Maxwell's six equations in ( ) involves the electric field component E_x:

$$\frac{\partial}{\partial t}\, E_x = \frac{1}{\epsilon}\left[\frac{\partial}{\partial y}\, H_z - \frac{\partial}{\partial z}\, H_y\right]. \tag{35}$$

Yee's difference equation computes E_x at the new time $(n+1)\Delta t$ from E_x at time $n\Delta t$ and the space differences of H_z and H_y **at time** $\boldsymbol{(n+\frac{1}{2})\Delta t}$. Figure 5.11 shows how those components of the magnetic field **H** are on a grid that is staggered with respect to the grid for **E**. We have six differential equations like (35) and six difference equations, to produce E_x, E_y, E_z at time $(n+1)\Delta t$ and then H_x, H_y, H_z at time $(n+1.5)\Delta t$.

The stability condition $c^2(\Delta t)^2 \leq (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ is acceptable. Perhaps the greatest drawback is the rectangular grid (finite elements are always more adaptable). But the FDTD method has been used with 10^9 meshpoints, which we cannot afford on an unstructured mesh. Finite differences also have numerical dispersion—the discrete wave speeds depend on the wave number $\boldsymbol{k} = (k_x, k_y, k_z)$. Those speeds don't exactly match c. We will have phase factors like F in equation ( ), extended to include Δy and Δz. When the dispersion creates significant errors, we can upgrade the spatial differences to fourth-order accuracy (using more mesh values). But those wider difference methods can go across material interfaces and external boundaries. This produces the ever-present give and take of numerical analysis: higher accuracy brings greater complexity. We can take larger steps Δt but every step is slower (and harder to code).

FIGURE TO COME...

Figure 5.11: $\dfrac{\Delta E}{\Delta t} = c\,\dfrac{\Delta H}{\Delta x}$ and $\dfrac{\Delta H}{\Delta t} = c\,\dfrac{\Delta E}{\Delta x}$ are staggered but centered.

Problem Set 5.3

1   Write the equation $u_{tt} = u_{xx} + u_{yy}$ as a first-order system $v_t = Av_x + Bv_y$ with the vector unknown $v = (u_t, u_x, u_y)$. The matrices A and B should be symmetric. Then the energy $E(t) = \frac{1}{2}\int (u_t^2 + u_x^2 + u_y^2)\,dx$ is *constant*.

2   How was the symmetry of A used in the final step $v^{\mathrm{T}}Av_x = (\frac{1}{2}v^{\mathrm{T}}Av)_x$ in equation (27)? You could write out $v^{\mathrm{T}}Av = \sum\sum a_{ij}v_i(x)v_j(x)$ and take the derivative of each term by the product rule.

3   Add Gauss law div $D = 0$ and div $B = 0$ with $D = \epsilon E$ and $B = \mu H$.