$Ly + \lambda f(x)\,y = 0$

Recall: If the Sturm-Liouville problem $\frac{d}{dx}[p(x)y'] + q(x)y + \lambda r(x)y = 0 \quad (a<x<b)$

+homogeneous boundary conditions at $x=a,b$

is "proper", then: $\cdot\ \lambda$ in $\{\lambda_n\}_{n=1,\dots\infty}$ eigenvalues

$\cdot\ y$ in $\{\phi_n(x)\}$

$\cdot$ "any" $f(x)$: $f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$ ①

To find a_n, we use the orthagonality relation

multiplies λ

$$\int_a^b dx\, r(x)\phi_n(x)\phi_m(x) = 0 \qquad \lambda_n \neq \lambda_m$$

$$① \cdot \overset{r(x)}{\phi_m(x)} \int_a^b r(x) f(x)\phi_m(x) \, dx = \sum_{n=1}^{\infty} a_n \int_a^b r(x)\phi_n(x)\phi_m(x)\,dx \quad {=0 \atop \text{unless } n=m}$$

$$\int_a^b dx\, r(x) f(x)\phi_m(x) = a_m \int_a^b r(x)[\phi_m(x)]^2 dx$$

$$\therefore a_m = \frac{\int_a^b dx\, r(x) f(x)\phi_m(x)}{\int_a^b dx\, r(x)[\phi_m(x)]^2}$$

Variation: $\quad Ly + \overset{\text{r fixed, given}}{\Lambda r(x)} y = h(x)$ — "source" $h \neq 0 \qquad a<x<b$

+homogeneous boundary conditions for y at $x = a, b$

seek $\quad y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \quad$ find $c_n \qquad L\phi_n + \lambda_n r(x)\phi_n = 0$

$$h(x) = Ly + \Lambda r(x) y = L\left[\sum c_n \phi_n(x)\right] + \Lambda r(x)\sum_n c_n\phi_n(x)$$

$$= \sum_n c_n \underbrace{L\phi_n(x)}_{-\lambda_n r\phi_n} + \Lambda r(x)\sum_n c_n\phi_n(x)$$

$$= \sum_n c_n r(x)\phi_n(x)(\Lambda - \lambda_n)$$

$$\underbrace{\frac{h(x)}{r(x)}}_{f(x)} = \sum_n \underbrace{\boxed{c_n(\Lambda-\lambda_n)}}_{a_n}\phi_n(x) \qquad r(x) \neq 0$$

$$c_n(\Lambda-\lambda_n) = a_n = \frac{\int_a^b dx\, \cancel{r(x)}\frac{h(x)}{\cancel{r(x)}}\phi_n(x)}{\int_a^b dx\, r(x)[\phi_n(x)]^2} = \frac{\int_a^b dx\, h(x)\phi_n(x)}{\int_a^b dx\, r(x)[\phi_n(x)]^2}$$

(i) $\Lambda \neq \lambda_n$ any $n \quad \rightarrow c_n = \frac{1}{\Lambda - \lambda(n)} a_n$

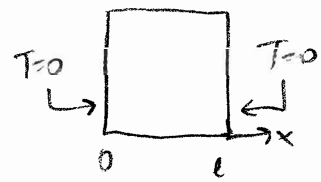
(ii) $\Lambda = \lambda_p \quad n=p \quad \rightarrow c_p \cdot 0 = \frac{\int_a^b dx\, h(x)\phi_p(x)}{\underbrace{\int_a^b dx\, r(x)\phi_p(x)^2}_{\neq 0}}$

- impossible if $\int_a^b dx\, h(x)\phi_p(x) \neq 0$
- has solution if $\quad$ " $\quad = 0$

Diffusion Equation

$$\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad (\alpha > 0),\ 0 < x < \ell$$

$T = T(x,t)$



$$T(x=0,t) = 0 = T(\ell, t)$$
$$T(x, t=0) = f(x)$$

- seek $T(x,t) = X(x)\,Q(t)$

PDE: $\dfrac{X''}{X} = \dfrac{1}{\alpha^2}\dfrac{Q'(t)}{Q(t)} = \text{constant} = -\lambda^2 < 0 \quad (\lambda > 0)$

- $\left.\begin{array}{l} X = A\cos(\lambda x) + B\sin(\lambda x) \\ X(0) = 0 = X(\ell) \end{array}\right\} \rightarrow X = A\sin\left(\frac{n\pi x}{\ell}\right) \qquad \lambda = \frac{n\pi}{\ell},\ n = 1, 2, \ldots$
- $Q(t) = Ce^{-\left(\frac{n\pi}{\ell}\right)^2 \alpha^2 t}$

$$T(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 \alpha^2 t} \longleftarrow \text{plug in}$$

Find D_n from initial conditions $(t=0)$: $\quad f(x) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right)$

$$D_n = \frac{2}{\ell}\int_0^{\ell} dx\, f(x) \sin\left(\frac{n\pi x}{\ell}\right)$$