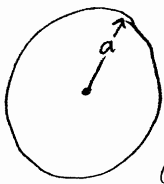
$$\nabla^2 u + k^2 u = 0$$
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$1\text{-D} = \frac{d^2u}{dx} + k^2 u = 0$$
$$2\text{-D} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u + k^2 u = 0$$ vibration of a 2-D surface



$u=0$ at $r=a$ (boundary condition) $u\Big|_{\sqrt{x^2+y^2}} = 0$

$u = u(x,y)$

(drum) use polar coordinates $r = \sqrt{x^2+y^2}$, $\theta = \arctan\frac{y}{x}$

$x = r\cos\theta$, $y = r\sin\theta$

Chain Rule: $\left(\frac{df}{dx} = \frac{dt}{dx}\frac{df}{dt}\right)$

$$\frac{d}{da}\arctan = \frac{1}{1+a^2}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \frac{1}{\cancel{2}}(x^2+a^2)^{-\frac{1}{2}}\cancel{2}x + \underline{\text{algebra}}$$

$$= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta\frac{\partial}{\partial r} + \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial r}{\partial x} = \cos\theta \quad \frac{\partial r}{\partial y} = \sin\theta \quad \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r} \quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \qquad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2 \qquad \frac{\partial^2}{\partial y} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right] - \sin\theta\cos\theta\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left[\cos\theta\frac{\partial}{\partial r}\right]$$

$$\frac{\partial^2 u}{\partial u^2} = \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2}\left(\cos\theta\frac{\partial u}{\partial\theta} + \sin\theta\frac{\partial^2 u}{\partial\theta^2}\right) - \sin\theta\cos\theta\left(\frac{-1}{r^2}\frac{du}{d\theta} + \frac{1}{r}\frac{\partial^2 u}{dr\,\partial\theta}\right)$$

$$+ \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial\theta\,\partial r}\right]$$

$x \to y$

$\theta \to \frac{\pi}{2} - \theta$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2}\left(-\sin\theta\frac{\partial u}{\partial\theta} + \cos\frac{\partial^2 u}{\partial\theta^2}\right) - \cos\theta\sin\theta\left(\frac{1}{r^2}\frac{\partial y}{\partial\theta} - \frac{1}{r}\frac{\partial^2 y}{\partial r\partial\theta}\right) - \frac{\cos\theta}{r}\left(-\cos\theta\frac{\partial u}{dr} - \sin\theta\frac{\partial^2 u}{\partial\theta\partial r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = -k^2 u$$

separating variables:

$$u(r,\theta) = R(r)e^{in\theta}$$

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = -k^2 R(r)$$

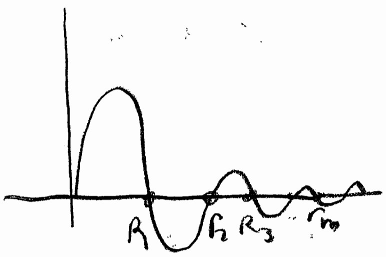
$$\frac{d^2R}{dr^2} = \frac{1}{r}J_n(kn) + Y_n(k_n)$$ Bessel equation

$R(n) = AJ_n(kr) + BY_n(kr)$ $\quad Y_n(0)\ \infty \rightarrow B=0$

$R(r) = AJ_n(kr)$ $\quad R(a) = 0 \rightarrow AJ_n(ka)=0$: root of bessel function

either $A=0$ trivial or $J_n(ka)=0$

$k_m = \frac{r_m}{a}$: eigenvalue



$$\boxed{J(n)\left(\frac{r_m}{a}r\right)e^{an\theta}}$$

$$\nabla u = -\left(\frac{r_m}{a}\right)^2 u$$
$$k^2 m$$

$$\frac{\Phi''}{\Phi} = -\lambda^2$$

$$\boxed{u = \sum_n \sum_m A_{nm} J_n\left(\frac{r_{nm}}{a} r\right) e^{in\theta}}$$

PDE: $\nabla^2\psi + k^2\psi = 0$, polar coordinates (ρ,ϕ), $0<\rho<a$, $0 \le \phi < 2\pi$

boundary conditions: $\psi|_{\rho=0}$: finite, $\psi|_{\rho=a} = 0$

try $\psi(\rho,\phi) = R(\rho)\cdot\Phi(\phi)$ separation of variables

Bessel ODE: for $R(\rho)$: $\rho^2 R''(\rho) + \rho R'(\rho) + [(k\rho)^2 - n^2]R = 0$

$\Phi''(\varphi) + \lambda^2\Phi(\varphi) = 0$

boundary conditions: $R(0)=$ finite, $R(a)=0$

$R(\rho) = cJ_n(k_m^{(n)}\rho)$

$k_m^{(n)} \equiv \frac{\gamma_m^{(n)}}{a}$ zeros of $J_n(x)$

boundary conditions: $\Phi(\varphi+2\pi) = \Phi(\varphi)$ periodicity of Φ

$\downarrow$

$\Phi(\varphi) = A\cos(n\varphi) + B_n\sin(n\varphi)$, $\lambda = n = 0, 1, \ldots$

Periodicity: $\Phi'(0) = \Phi'(2\pi)$, $\Phi(0) = \Phi(2\pi)$

$$\Phi(\varphi) = A\cos(\lambda\varphi) + B\sin(\lambda\varphi)$$

$$\left.\begin{aligned} A &= A\cos(-\lambda 2\pi) + B\sin(\lambda 2\pi) \\ \lambda B &= -\lambda A\sin(\lambda 2\pi) + \lambda B\cos(\lambda \cdot 2\pi) \end{aligned}\right\} \xrightarrow{A \neq \text{ or } B \neq 0} \det = 0 \rightarrow \lambda = n$$