ex Solve $x y'' + y' + xy = 0$ $\quad y(0)$: finite, $y(1)=1$ $\quad 0<x<1$

$$x^2 y'' + xy' + (x^2 - 0)y = 0 \; : \; p = 0$$

$$y(x) = c_1 J_0(x) + c_2 Y_0(x)$$ $c_2 = 0$ because Y_0 blows up at 0

Find c_1: $\quad y(1) = 1 \longleftrightarrow c_1 J_0(1) = 1 \longleftrightarrow c_1 = \frac{1}{J_0(1)}$

$$\boxed{y(x) = \frac{J_0(x)}{J_0(1)}}$$ unique!

Application: Solving ODEs other than Bessel's equation.

Bessel equation: $X^2 Y''(X) + X Y'(X) + (X^2 - p^2) Y(X) = 0$

change of variable: $Y = \frac{y}{g(x)}$, $X = f(x)$ $\quad (X, Y) \rightarrow (x, y)$

means linear combination of Bessel functions

general solution: $Y(X) = Z_p(X) \rightarrow y(x) = g(x) Y(X) = \boxed{g(x) Z_p(f(x))}$

ODE for $y(x)$: $f \frac{d}{dx}\left[\frac{\frac{d}{dx}\left(\frac{y}{g}\right)}{f'(x)}\right] + \frac{d}{dx}\left(\frac{y}{g}\right) + \frac{f'}{f}(f^2 - p^2)\frac{y}{g} = 0$

specialize: $f(x) = Cx^s$, $g(x) = x^A e^{-Bx^r}$

ODE: $x^2 \frac{d^2 y}{dx^2} + x\left[(1-2A) + 2rBx^r\right]\frac{dy}{dx} + \left[A^2 - p^2 s^2 + s^2 C^2 x^{2s} - rB(2A - r)x^r + r^2 B^2 x^{2r}\right] y = 0$

$$\boxed{y(x) = x^A e^{-Bx^r} Z_p(C \cdot x^s)}$$

ex $xy''-3y'+xy=0$

$x^2y''-3xy'+x^2y=0$

coefficient of $\frac{dy}{dx}$: $-3x = x(1-2A)+2rBx^{r+1}$ find A, B, r

$1-2A=-3$ $r=0$ or $B=0$ without loss of generality

$A=2$

coefficient of y: $x^2 = A^2-p^2s^2+s^2C^2x^{2s}$ $s=1$ $C=1$

$A^2-p^2s^2=0$ $p=2$ ← p is nonnegative

$$y(x)=x^2 Z_2(x)$$

ex $x^4y''+a^2y=0$ $a>0$

$x^2y''+\frac{a^2}{x^2}y=0$

y': $1-2A+2rBx^r=0$ $A=\frac{1}{2}$ B or $r=0$

y: coeff $=a^2x^{-2}=A^2-p^2s^2+s^2C^2x^{2s}$ $s=-1$

$A^2-p^2s^2=0$ $p^2=\frac{A^2}{s^2}=\frac{1}{4}$ $p=\frac{1}{2}$

$C=a>0$

$$y(x)=\sqrt{x}\, Z_{\frac{1}{2}}\left(\frac{a}{s}\right)=\sqrt{x}\left\{c_1 \underbrace{J_{1/2}\left(\frac{a}{x}\right)}_{\sqrt{\frac{x2}{\pi a}}\sin\left(\frac{a}{x}\right)}+c_2 \underbrace{J_{-\frac{1}{2}}\left(\frac{a}{x}\right)}_{\sqrt{\frac{2x}{\pi a}}\cos\left(\frac{a}{x}\right)}\right\}$$

ex $xy''+(1+2x)y'+y=0$

$x^2y''+(x+2x^2)y'+xy=0$

y': $x(1+2x)=x([1-2A]+2rBx^r]$ $1-2A=1 \rightarrow A=0$

$2rBx^r=2x \rightarrow r=1$ $B=1$

y: $x=-p^2s^2+s^2C^2x^{2s}+x+x^2$ $s=1$ $C=i$ $p=0$

$$y(x)=e^{-x}Z_0(ix)=e^{-x}(c_1 I_0(x)+c_2 K_0(x))$$

modified Bessel functions