Bessel Equation: $x^2 y'' + xy' + (x^2 - p^2)y = 0$

$\underline{x_0 = 0}$: regular singular $y'' + \frac{1}{x}y' + \frac{x^2-p^2}{x^2}y = 0$

$R(x) = 1 \qquad R_M = 0$

$P(x) = 1 \qquad P_0 = 1,\ P_M = 0$

$Q(x) = -p^2 + x^2 \qquad Q_0 = -p^2,\ Q_M = 1$

$$f(s) = s(s-1) + s - p^2 = 0 \qquad (p \geq 0)$$

$$s^2 \cancel{- s + s} - p^2 = 0$$

$$s = \pm p$$

Frobenius gives 2 independent solutions if $s_1 - s_2 = 2p \neq$ integer

$$y(x) = \sum_{l=0}^{\infty} B_l \cdot x^{2l+s} \qquad (M=2) \qquad k = 2l \quad \text{(look at previous page)}$$

$$f(s+2l)\underbrace{B_l}_{A_k} + g(s+2l)\underbrace{B_{l-1}}_{A_{k-2}} = 0$$

$$g(s) = \cancel{R_m(s-m-1)(s-m)}^{0} + \cancel{P_m(s-m)}^{0} + \overset{1}{Q_M} = 1$$

$$\underbrace{(s+p+2l)(s+2l-p)}_{f(s+2l) = (s+2l)^2 - p^2} B_l = -B_{l-1}$$

$$s = s_1 = p$$

$$\boxed{B_l = -\frac{B_{l-1}}{(s+p+2l)(s+2l-p)}} \quad \text{recurrence relation } l = 1, \dots, \infty$$

$\underline{s_1 = p}$: $B_l = \dfrac{(-1)^l B_0}{(2^l l!)(2+2p)(4+2p)(6+2p)\dots(2l+2p)}$

$$= \frac{(-1)^l B_0}{(2^{2l} l!)(1+p)(2+p)\dots(l+p)} \cdot \frac{\Gamma(1+p)}{\Gamma(1+p)}$$

$\Gamma(2+p)$ $\Gamma(3+p)$

Digression: Gamma function

$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}$, $\operatorname{Re} z > 0$

- $\Gamma(n+1) = n!$ (n integer)
- $\Gamma(z+1) = z\Gamma(z)$ (any z)

$$y(x) = x^p \sum_{l=0}^{\infty} B_l x^{2l} = B_0 \Gamma(1+p) \cdot 2^p \boxed{\sum_{l=0}^{\infty} \frac{(-1)^l \left(\frac{x}{2}\right)^{2l+p}}{l!\, \Gamma(l+p+1)}}$$

$J_p(x)$: Bessel function of order p.

Bessel function of order p: $J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k!\,\Gamma(k+p+1)}$ $(p \geq 0)$

$J_p(x)$ is 1 of 2 solutions of Bessel equation: $x^2 y'' + xy' + (x^2 - p^2) y = 0$

$s_1 = p$ $\quad s_2 = -p$ $\quad s_1 - s_2 = 2p$ (not an integer) $\rightarrow$ 2 indep solutions $J_p(x)$, $J_{-p}(x)$

$s_1 = p$: $\quad y_1(x) = c_1 J_p(x)$

$s_2 = -p$: $\quad y_2(x) = c_2 J_{-p}(x)$ $\qquad J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-p}}{k!\,\Gamma(k-p+1)}$

We can prove (4.8) that J_p and J_{-p}' are independent solutions when $p \neq$ integer.

Digression: $\Gamma(z) = \int_0^{\infty} dt\, e^{-t} t^{z-1}$, $\quad \mathrm{Re}\, z > 0$ $\qquad (t^{z-1} > 0)$

$\Gamma(z)$: function of the complex variable z

- has only simple poles in complex plane at $z = -n$, $n = 0, 1, 2, \ldots$

- If $p \neq$ integer, general solution of Bessel's equation: $y(x) = c_1 J_p(x) + c_2 J_{-p}(x)$
- p = integer = n

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!\,\Gamma(k+n+1)} \qquad (n \geq 0)$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\,\Gamma(k-n+1)} \qquad k-n+1 \leq 0 \rightarrow k \leq n-1$$

$$\rightarrow \Gamma(k-n+1) \rightarrow \infty$$

$$= \sum_{n}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\,\Gamma(k-n+1)} \rightarrow m = k-n \,;\ k = m+n$$

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (x/2)^{2m+n}}{\underbrace{(m+n)!}_{\Gamma(m+n+1)}\,\underbrace{\Gamma(m+1)}_{m!}} = (-1)^n \underbrace{\boxed{\sum_{m=0}^{\infty} (-1)^m \frac{(x/2)^{2m+n}}{\Gamma(m+n+1)\, m!}}}_{J_n(x)} \qquad J_{-n}(x) = (-1)^n J_n(x)$$

F.S. gives only one solution when $p = n$ = integer.

We seek a 2nd solution in form

$$y_2(x) = C(\ln x)\underbrace{J_n(x)}_{y_1(x)} + \overbrace{x^{-n}\sum_{k=0}^{\infty} B_k x^k}^{S_2} \qquad \text{find } C,\ B_k \ \text{(functions of } B_0)$$

$\underline{p=n=0:}$ $$y_2(x) = B_0 \underbrace{\left[(\ln x)\, J_0(x) + \sum_{k=1}^{\infty} (-1)^{k+1} \phi(k) \frac{(x/2)^{2k}}{(k!)^2}\right]}_{Y^{(0)}(x)} \qquad \begin{cases} \phi(k) = 1 + \frac{1}{2} + \dots + \frac{1}{k} \\ \phi(0) = 0 \end{cases}$$

$$Y_0(x) = \frac{2}{\pi}\left[Y^{(0)}(x) + (\gamma + \ln 2) J_0(x)\right] \leftarrow \text{Neumann function of order } 0$$

γ: Euler's constant: 0.577...

$$\rightarrow Y_0(x) = \frac{2}{\pi}\left\{\left(\ln\frac{x}{2} + \gamma\right) J_0(x) + \sum_{k=0}^{\infty} (-1)^{k+1} \phi(k) \frac{(x/2)^{2k}}{(k!)^2}\right\}$$

$\underline{p=n:}$ $$Y_n(x) = \frac{2}{\pi}\left\{\left(\ln\frac{x}{2} + \gamma\right) J_n(x) - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\,(x/2)^{2k-n}}{k!} + \frac{1}{2}\sum_{k=0}^{\infty} (-1)^{k+1}\left[\phi(k) + \phi(k+n)\right]\frac{(x/2)^{2k+n}}{k!\,(k+n)!}\right\}$$

General solution of Bessel's equation for $p=n$:

$$\boxed{y(x) = c_1 J_n(x) + c_2 Y_n(x)}$$

* Bessel's equation $x^2 y'' + x y' + (x^2 - p^2) y = 0$ has general solution:

$$y(x) = \begin{cases} c_1 J_p(x) + c_2 J_{-p}(x) & p \neq n \\ c_1 J_n(x) + c_2 Y_n(x) & p = n \end{cases} \qquad y(x) \equiv Z_p(x)$$

Can define $Y_p(x)$, $p \neq n$: $Y_p(x) = \dfrac{\cos(p\pi) J_p(x) - J_{-p}(x)}{\sin(p\pi)}$

if $p=n$: $Y_n(x)$

For any p, $y(x) = c_1 J_p(x) + c_2 Y_p(x)$