IV Other Integrals

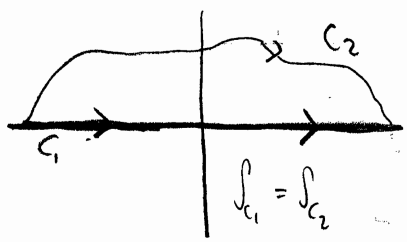
ex $\quad I = \int_0^\infty dx \, \frac{\sin x}{x} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}$ ← even

Method 1

1. let $x \to z$: $f(z) = \frac{\sin(z)}{z}$: analytic everywhere.
2. close the path.

$$\cancel{I = \frac{1}{2}\int_{-\infty}^{\infty} dx \, \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\frac{1}{x} = \frac{1}{4i}\left[\int_{-\infty}^{\infty} \frac{e^{ix}}{x} - \int_{-\infty}^{\infty} dx \, \frac{e^{-ix}}{x}\right]}$$

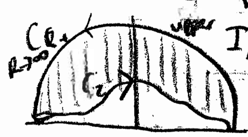
~~divergent at $x=0$~~



$$I = \frac{1}{2}\int_{C_2} dz \, \frac{\sin z}{z}$$

$$= \frac{1}{4i}\left[\underbrace{\int_{C_2} dz \, \frac{e^{iz}}{z}}_{I_A} - \underbrace{\int_{C_2} dz \, \frac{e^{-iz}}{z}}_{I_B}\right]$$

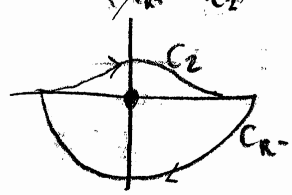
- $I_A = \int_{C_2} dz \, \frac{e^{iz}}{z} = 0$



by Cauchy Integral Formula

$C_+ = C_2 + C_{R+}$: $\oint_{C+} dz \, \frac{e^{iz}}{z} = 0 = \left(\cancel{\int_{C_{R+}}}^{\,0} + \int_{C_2}\right) dz \, \frac{e^{iz}}{z} \to \boxed{I_A = 0}$

- $I_B = \int_{C_2} dz \, \frac{e^{-iz}}{z}$

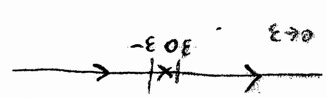


simple pole at 0

$C_- = C_2 + C_{R-}$

Residue theorem: $\oint_C dz \, \frac{e^{iz}}{z} = -2\pi i \cdot \frac{e^{i0}}{1} = -2\pi i \quad \boxed{I_B = -2\pi i}$

$\frac{1}{4\cancel{i}} \cdot 2\pi \cancel{i} = \boxed{\frac{\pi}{2}}$

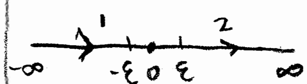


Alternatively, $I = \frac{1}{2}\lim_{\varepsilon \to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty}\right) \frac{\sin x}{x} \, dx$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

Method 2: $$I = \frac{1}{2}\lim_{\varepsilon\to 0}\left(\int_1 + \int_2\right)\frac{\sin x}{x}\,dx = \frac{1}{2}\underbrace{\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{+\varepsilon}^{+\infty}\right)\frac{\sin x}{x}\,dx}_{\equiv P\int_{-\infty}^{\infty} dx \frac{\sin x}{x}}$$

principal value for $x=0$ of $\int_{-\infty}^{\infty} dx \frac{\sin x}{x}$



doesn't make any sense without the P

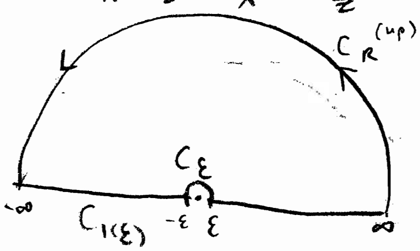
$$I = \frac{1}{2} P\int_{-\infty}^{\infty} dx \frac{\sin x}{x} = \frac{1}{2}\underbrace{\operatorname{Im} P\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x}}$$

$\frac{\sin x}{x}$: $= \operatorname{Im} e^{ix}$

$$\lim_{\varepsilon\to 0}\operatorname{Im}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)dx\,\frac{e^{ix}}{x} \quad \text{: well-defined}$$

$$\underbrace{\lim_{\varepsilon\to 0}\operatorname{Im}}_{\operatorname{Im}\lim_{\varepsilon\to 0}}$$

$x \to z$; $\frac{e^{ix}}{x} \to \frac{e^{iz}}{z}$



$(e^{i\alpha z},\ \alpha = 1 > 0)$

$$C = C_\varepsilon + C_{1(\varepsilon)} + C_R^{(up)}$$

Residue Theorem: $$\oint_C dz \frac{e^{iz}}{z} = 0 = \Big(\int_{C_\varepsilon} + \int_{C_1} + \underbrace{\cancel{\int_{C_R^{(up)}}}}_{\to 0}\Big) dz\, \frac{e^{iz}}{z}$$

$$\cdot \left|\int_{C_R} dz \frac{e^{iz}}{z}\right| = \left|\int_0^{\pi} i\cancel{Re^{i\theta}}\,d\theta\,\frac{e^{iR(\cos\theta + i\sin\theta)}}{\cancel{Re^{i\theta}}}\right| \le \int_0^{\pi} d\theta\, e^{-R\sin\theta}$$

positive

$z = Re^{i\theta}$

$dz = iRe^{i\theta}d\theta$

$$= \int_0^{\pi/2} d\theta\, e^{-R\sin\theta} + \int_{\pi/2}^{0} d\theta\, e^{-R\sin\theta} = \int_0^{\frac{\pi}{2}} d\theta\, e^{-R\sin\theta} + \int_{-\pi/2}^{0} d\psi\, e^{R\sin\psi}$$

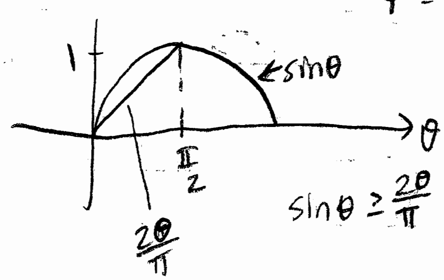
$\psi = \theta - \pi$

$\psi \to -\psi$

$$\int_0^{\frac{\pi}{2}} d\psi\, e^{-R\sin\psi}$$

$$= 2\int_0^{\pi/2} d\theta\, e^{-R\sin\theta}$$

$\sin\theta \to \frac{2\theta}{\pi}$



$\sin\theta \ge \frac{2\theta}{\pi}$

$$\le 2\int_0^{\frac{\pi}{2}} d\theta\, e^{-R\frac{2\theta}{\pi}}$$

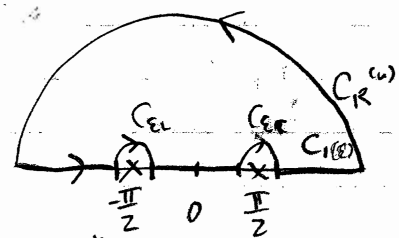
$$= 2\frac{\pi}{2R}(1 - e^{-R})$$: integral goes to 0 not exponentially, but as $\frac{1}{R}$

$$\bullet \int_{C_1(\varepsilon)} dz\, \frac{e^{iz}}{z} = \int_{\pi}^{0} \cancel{i\varepsilon e^{i\varphi}}\, d\varphi\, \frac{e^{i\varepsilon e^{i\varphi}}}{\cancel{\varepsilon e^{i\varphi}}} = -i\int_0^{\pi} d\varphi\, e^{i\varepsilon e^{i\varphi}} \xrightarrow[\varepsilon\to 0]{} -i\int_0^{\pi} d\varphi = -i\pi$$

$z = \varepsilon e^{i\varphi}$

$$0 = -i\pi + \int_{\substack{C_1(\varepsilon)\\(\varepsilon\to 0)}} dz\, \frac{e^{iz}}{z} \rightarrow \boxed{\int_{\substack{C_1(\varepsilon)\\(\varepsilon\to 0)}} dz\, \frac{e^{iz}}{z} = i\pi}$$

$$I = \frac{1}{2}\,\mathrm{Im} \int_{\substack{C_{1(\varepsilon)}\\(\varepsilon\to 0)}} dz\, \frac{e^{iz}}{z} = \boxed{\frac{\pi}{2}}$$

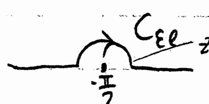


<u>ex</u> $I = P\int_{-\infty}^{\infty} dx\, \frac{\cos x}{x^2 - \pi^2/4}$ $\mathrm{Re}(e^{ix})$

$$= \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\frac{\pi}{2}-\varepsilon} + \int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}+\varepsilon} + \int_{\frac{\pi}{2}+\varepsilon}^{\infty}\right) dx\, \frac{\cos x}{x^2-\pi^2/4}$$

$$= \mathrm{Re}\; P\int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{x^2-\pi^2/4}$$

$$I_\varepsilon = \int_{C_1(\varepsilon)} dz\, \frac{e^{iz}}{z^2+\pi^2/4} \qquad \oint_C dz\, \frac{e^{iz}}{z^2-\pi^2/4} = 0 = \left(\int_{C_{\varepsilon l}} + \int_{C_{\varepsilon r}} + \int_{C_1} + \cancelto{0}{\int_{C_R}}\right) dz\, \frac{e^{iz}}{z^2-\pi^2/4} \qquad \begin{matrix} R\to\infty \\ \varepsilon\to 0\end{matrix}$$



$z + \frac{\pi}{2} = \varepsilon e^{i\varphi}$ $\qquad dz = i\varepsilon e^{i\varphi} d\varphi$

$$\int_{C_{\varepsilon l}} dz\, \frac{e^{iz}}{z^2-\pi^2/4} = \lim_{\varepsilon\to 0}\int_{\pi}^{0} \cancel{\varepsilon} i \cancel{e^{i\varphi}}\, d\varphi\, \frac{e^{i(\cancelto{0}{\varepsilon e^{i\varphi}}-\frac{\pi}{2})}}{(\cancelto{0}{\varepsilon e^{i\varphi}}-\pi)\cancel{\varepsilon e^{i\varphi}}} = i\, e^{-i\pi/2}\, \frac{1}{\cancel{-\pi}}\,(\cancel{-\pi}) = 1$$

$(z-\frac{\pi}{2})(z+\frac{\pi}{2})$ $\qquad \lim_{\varepsilon\to 0}\int_{C_{\varepsilon r}} dz\, \frac{e^{iz}}{z^2-\pi^2/4} = 1$

$$\int_{\substack{C_{1(\varepsilon)}\\ \varepsilon\to 0}} dz\, \frac{e^{iz}}{z^2-\pi^2/4} = -2\,, \qquad \boxed{I = -2}$$

$$\frac{1}{x^2-\pi^2/4} = \left(\frac{1}{x-\pi/2} - \frac{1}{x+\pi/2}\right)\frac{1}{\pi}$$

$$\int dx\, \frac{\cos x}{x^2-\pi/4} = \frac{1}{\pi}\left[\int_{-\infty}^{\infty} dx\, \frac{\cos x}{x-\pi/2} + \int_{-\infty}^{\infty} dx\, \frac{\cos x}{x+\pi/2}\right]$$

let $x - \pi/2 = y$ $\qquad$ let $x + \frac{\pi}{2} = y$

$\downarrow$

$\frac{\sin y}{y}$ $\qquad$ $\frac{\sin y}{y}$