NAME:

18.075 In–class Exam # 2

November 3, 2004

Answer all questions. Justify your answers. Cross out what is not meant to be part of your final answer. Total number of points:67.5. Extra 5 points may be given as bonus.

I. Consider the following integral:

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{(4x^2 - 9\pi^2)(x^2 + 9)}\, dx.$$

1.(2 pts) Locate and characterize all singularities of the integrand, $\frac{\cos z}{(4z^2-9\pi^2)(z^2+9)}$, in the complex z plane.

2.(2 pts) Locate and characterize all singularities of $\frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)}$, i.e., repeat part (1) above by replacing $\cos x$ by e^{ix} (and x by the complex variable z).

3.(6 pts) Define contours in order to evaluate the given integral I. Give the definition of the required principal value as the appropriate integral of $\frac{e^{ix}}{(4x^2-9\pi^2)(x^2+9)}$.

4.(10 pts) Evaluate the integral I by use of the residue theorem.

II. Find the region of convergence of the following power series, using either the ratio test or the root test.

1.(5 pts)

$$\sum_{n=1}^{\infty} \frac{2^n}{n^n} z^n,$$

2.(5 pts)

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{2^n}(z-1)^{3n}.$$

Hint: Use the ratio or root test for the entire term $A_n(x) = \frac{n(n+1)}{2^n}(z-1)^{3n}$.

III. Consider the integral

$$I = \int_0^\pi d\theta \, \frac{1}{2 + \sin^2\theta}.$$

1.(4 pts) Explain what change(s) of variable you need to make in order to transform this integral to one along the ENTIRE unit circle. **Hint:** You may use a trigonometric identity to simplify $\sin^2\theta$.

2.(10 pts) Evaluate the given integral by use of the residue theorem.

IV. Consider the ordinary differential equation (ode)

$$(x^2 - x)y'' - (x^2 + 1)y' - (x - 1)y = 0 \qquad (1).$$

1.(3 pts) Put this equation in the form $y'' + a_1(x)y' + a_2(x)y = 0$ and locate all its singular points.
Recall: Singular points in this case are those at which $a_1(z)$ or $a_2(z)$ is NOT analytic.

2.(3 pts) Classify the point $x_0 = 0$ (ordinary point, regular or irregular singular point ?).

3.(4 pts) Substitute the power series

$$y = \sum_{n=0}^{\infty} A_n x^n$$

into the given ode. Express the left-hand side of the ode as a power series with each term having the (common) factor x^n.

4.(9.5 pts) Derive the recurrence formula for A_n. From this formula, determine how many independent solutions this method gives. Give the first three nonvanishing term of each infinite series involved.

5. (4 pts) Put the ODE (1) of page 8 to the canonical form for $x_0 = 0$,

$$R(x)\, y'' + \frac{P(x)}{x}\, y' + \frac{Q(x)}{x^2} y = 0, \qquad (2)$$

where $R(z)$, $P(z)$ and $Q(z)$ are analytic at $z = x_0 = 0$, with $R(0) \neq 0$.

6. (BONUS, 5pts) <u>Write down</u> the indicial equation for ODE (2) and <u>determine</u> its roots. How many solutions would the Frobenius method give for this ODE ? (You don't need to find these solutions here.)