Class 8 Review Problems –solutions, 18.05, Spring 2014

1 Counting and Probability

1. **(a)** Create an arrangement in stages and count the number of possibilities at each stage:

Stage 1: Choose three of the 10 slots to put the S's: $\binom{10}{3}$

Stage 2: Choose three of the remaining 7 slots to put the T's: $\binom{7}{3}$

Stage 3: Choose two of the remaining 4 slots to put the I's: $\binom{4}{2}$

Stage 4: Choose one of the remaining 2 slots to put the A: $\binom{2}{1}$

Stage 5: Use the last slot for the C: $\binom{1}{1}$

Number of arrangements:

$$\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 50400.$$

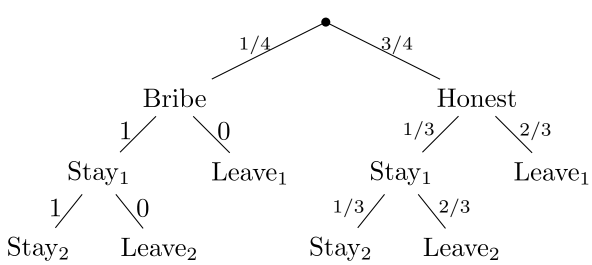
(b) The are $\binom{10}{2} = 45$ equally likely ways to place the two I's.

There are 9 ways to place them next to each other, i.e. in slots 1 and 2, slots 2 and 3, . . . , slots 9 and 10.

So the probability the I's are adjacent is $9/45 = 0.2$.

2 Conditional Probability and Bayes' Theorem

2. The following tree shows the setting. Stay_1 means the contestant was allowed to stay during the first episode and stay_2 means the they were allowed to stay during the second.



Let's name the relevant events:
B = the contestant is bribing the judges
H = the contestant is honest (not bribing the judges)
S_1 = the contestant was allowed to stay during the first episode
S_2 = the contestant was allowed to stay during the second episode

L_1 = the contestant was asked to leave during the first episode
L_2 = the contestant was asked to leave during the second episode

(a) We first compute $P(S_1)$ using the law of total probability.

$$P(S_1) = P(S_1|B)P(B) + P(S_1|H)P(H) = 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{2}.$$

We therefore have (by Bayes' rule) $P(B|S_1) = P(S_1|B)\dfrac{P(B)}{P(S_1)} = 1 \cdot \dfrac{1/4}{1/2} = \dfrac{1}{2}.$

(b) Using the tree we have the total probability of S_2 is

$$P(S_2) = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}$$

(c) We want to compute $P(L_2|S_1) = \dfrac{P(L_2 \cap S_1)}{P(S_1)}.$

From the calculation we did in part (a), $P(S_1) = 1/2$. For the numerator, we have (see the tree)

$$P(L_2 \cap S_1) = P(L_2 \cap S_1|B)P(B) + P(L_2 \cap S_1|H)P(H) = 0 \cdot \frac{1}{3} + \frac{2}{9} \cdot \frac{3}{4} = \frac{1}{6}$$

Therefore $\boxed{P(L_2|S_1) = \dfrac{1/6}{1/2} = \dfrac{1}{3}.}$

3 Independence

3. E = even numbered = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}.
L = roll ≤ 10 = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
B = roll is prime = {2, 3, 5, 7, 11, 13, 17, 19} (We use B because P is not a good choice.)

(a) $P(E) = 10/20$, $P(E|L) = 5/10$. These are the same, so the events are independent.

(b) $P(E) = 10/20$. $P(E|B) = 1/8$. These are not the same so the events are not independent.

4 Expectation and Variance

4. **(a)** We have

X values:	-1	0	1
prob:	1/8	2/8	5/8
X^2	1	0	1

So, $\boxed{E(X) = -1/8 + 5/8 = 1/2.}$

(b)

Y values:	0	1
prob:	2/8	6/8

$\Rightarrow \boxed{E(Y) = 6/8 = 3/4.}$

(c) The change of variables formula just says to use the bottom row of the table in part (a): $E(X^2) = 1 \cdot (1/8) + 0 \cdot (2/8) + 1 \cdot (5/8) = 3/4$ (same as part (b)).

(d) $\boxed{\text{Var}(X) = E(X^2) - E(X)^2 = 3/4 - 1/4 = 1/2.}$

5. Make a table

X:	0	1
prob:	(1-p)	p
X^2	0	1.

From the table, $\boxed{E(X) = 0 \cdot (1-p) + 1 \cdot p = p.}$

Since X and X^2 have the same table $E(X^2) = E(X) = p$.

Therefore, $\boxed{\text{Var}(X) = p - p^2 = p(1-p).}$

6. Let X be the number of people who get their own hat.

Following the hint: let X_j represent whether person j gets their own hat. That is, $X_j = 1$ if person j gets their hat and 0 if not.

We have, $X = \sum_{j=1}^{100} X_j$, so $E(X) = \sum_{j=1}^{100} E(X_j)$.

Since person j is equally likely to get any hat, we have $P(X_j = 1) = 1/100$. Thus, $X_j \sim$ Bernoulli$(1/100)$ $\Rightarrow$ $E(X_j) = 1/100$ $\Rightarrow$ $\boxed{E(X) = 1.}$

5 Probability Mass Functions, Probability Density Functions and Cumulative Distribution Functions

7. **(a)** We have cdf of X,

$$F_X(x) = \int_0^x \lambda \mathrm{e}^{-\lambda x} dx = 1 - \mathrm{e}^{-\lambda x}.$$

Now for $y \geq 0$, we have

(b)

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = 1 - \mathrm{e}^{-\lambda\sqrt{y}}.$$

Differentiating $F_Y(y)$ with respect to y, we have

$$f_Y(y) = \frac{\lambda}{2} y^{-\frac{1}{2}} \mathrm{e}^{-\lambda\sqrt{y}}.$$

8. **(a)** There are a number of ways to present this.

Let T be the total number of times you roll a 6 in the 100 rolls. We know $T \sim$ Binomial$(100, 1/6)$. Since you win \$3 every time you roll a 6, we have $X = 3T$. So, we can write

$$P(X = 3k) = \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}, \quad \text{for } k = 0,\ 1,\ 2,\ \ldots,\ 100.$$

Alternatively we could write

$$P(X = x) = \binom{100}{x/3} \left(\frac{1}{6}\right)^{x/3} \left(\frac{5}{6}\right)^{100-x/3}, \quad \text{for } x = 0, 3, 6, \ldots, 300.$$

(b) $E(X) = E(3T) = 3E(T) = 3 \cdot 100 \cdot \frac{1}{6} = 50$,
$\text{Var}(X) = \text{Var}(3T) = 9\text{Var}(T) = 9 \cdot 100 \cdot \frac{1}{6} \cdot \frac{5}{6} = 125$.

(c) (i) Let T_1 be the total number of times you roll a 6 in the first 25 rolls. So, $X_1 = 3T_1$ and $Y = 12T_1$.
Now, $T_1 \sim \text{Binomial}(25, 1/6)$, so

$$E(Y) = 12E(T_1) = 12 \cdot 25 \cdot 16 = 50.$$

and

$$\text{Var}(Y) = 144\text{Var}(T_1) = 144 \cdot 25 \cdot \frac{1}{6} \cdot \frac{5}{6} = 500.$$

(ii) The expectations are the same by linearity because X and Y are the both
$3 \times 100 \times$ a Bernoulli(1/6) random variable.

For the variance, $\text{Var}(X) = 4\text{Var}(X_1)$ because X is the sum of 4 *independent* variables all identical to X_1. However $\text{Var}(Y) = \text{Var}(4X_1) = 16\text{Var}(X_1)$. So, the variance of Y is 4 times that of X. This should make some intuitive sense because X is built out of more independent trials than X_1.

Another way of thinking about it is that the difference between Y and its expectation is four times the difference between X_1 and its expectation. However, the difference between X and its expectation is the sum of such a difference for X_1, X_2, X_3, and X_4. Its probably the case that some of these deviations are positive and some are negative, so the absolute value of this difference for the sum is probably less than four times the absolute value of this difference for one of the variables. (I.e., the deviations are likely to cancel to some extent.)

6 Joint Probability, Covariance, Correlation

9. (**Arithmetic Puzzle**) **(a)** The marginal probabilities have to add up to 1, so the two missing marginal probabilities can be computed: $P(X = 3) = 1/3$, $P(Y = 3) = 1/2$. Now each row and column has to add up to its respective margin. For example, $1/6 + 0 + P(X = 1, Y = 3) = 1/3$, so $P(X = 1, Y = 3) = 1/6$. Here is the completed table.

$X \backslash Y$	1	2	3	
1	1/6	0	1/6	1/3
2	0	1/4	1/12	1/3
3	0	1/12	1/4	1/3
	1/6	1/3	1/2	1

(b) No, X and Y are not independent.
For example, $P(X = 2, Y = 1) = 0 \neq P(X = 2) \cdot P(Y = 1)$.

10. Covariance and Independence

(a)

X / Y	-2	-1	0	1	2	
0	0	0	1/5	0	0	1/5
1	0	1/5	0	1/5	0	2/5
4	1/5	0	0	0	1/5	2/5
	1/5	1/5	1/5	1/5	1/5	1

Each column has only one nonzero value. For example, when $X = -2$ then $Y = 4$, so in the $X = -2$ column, only $P(X = -2, Y = 4)$ is not 0.

(b) Using the marginal distributions: $E(X) = \frac{1}{5}(-2 - 1 + 0 + 1 + 2) = 0.$

$E(Y) = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 4 \cdot \frac{2}{5} = 2.$

(c) We show the probabilities don't multiply:

$P(X = -2,\ Y = 0) = 0 \neq P(X = -2) \cdot P(Y = 0) = 1/25.$

Since these are not equal X and Y are not independent. (It is obvious that X^2 is not independent of X.)

(d) Using the table from part (a) and the means computed in part (d) we get:

$$\begin{aligned}\mathrm{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ &= \frac{1}{5}(-2)(4) + \frac{1}{5}(-1)(1) + \frac{1}{5}(0)(0) + \frac{1}{5}(1)(1) + \frac{1}{5}(2)(4) \\ &= 0.\end{aligned}$$

11. Continuous Joint Distributions

(a) $F(a, b) = P(X \le a, Y \le b) = \int_0^a \int_0^b (x + y)\, dy\, dx.$

Inner integral: $xy + \left.\frac{y^2}{2}\right|_0^b = xb + \frac{b^2}{2}$. Outer integral: $\left.\frac{x^2}{2}b + \frac{b^2}{2}x\right|_0^a = \frac{a^2 b + ab^2}{2}.$

So $\boxed{F(x, y) = \frac{x^2 y + xy^2}{2}}$ and $\boxed{F(1, 1) = 1.}$

(b) $f_X(x) = \int_0^1 f(x, y)\, dy = \int_0^1 (x + y)\, dy = xy + \left.\frac{y^2}{2}\right|_0^1 = \boxed{x + \frac{1}{2}.}$

By symmetry, $\boxed{f_Y(y) = y + 1/2.}$

(c) To see if they are independent we check if the joint density is the product of the marginal densities.

$f(x, y) = x + y, \quad f_X(x) \cdot f_Y(y) = (x + 1/2)(y + 1/2).$

Since these are not equal, $\boxed{X \text{ and } Y \text{ are not independent.}}$

(d) $E(X) = \int_0^1 \int_0^1 x(x + y)\, dy\, dx = \int_0^1 \left[\left. x^2 y + x\frac{y^2}{2}\right|_0^1\right] dx = \int_0^1 x^2 + \frac{x}{2}\, dx = \boxed{\frac{7}{12}.}$

(Or, using (b), $E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(x+1/2)\,dx = 7/12.$)

By symmetry $\boxed{E(Y) = 7/12.}$

$E(X^2+Y^2) = \int_0^1 \int_0^1 (x^2+y^2)(x+y)\,dy\,dx = \boxed{\frac{5}{6}.}$

$E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dy\,dx = \boxed{\frac{1}{3}.}$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \frac{49}{144} = \boxed{-\frac{1}{144}.}$

7 Law of Large Numbers, Central Limit Theorem

12. Standardize:

$$\begin{aligned} P\left(\sum_i X_i < 30\right) &= P\left(\frac{\frac{1}{n}\sum X_i - \mu}{\sigma/\sqrt{n}} < \frac{30/n - \mu}{\sigma/\sqrt{n}}\right) \\ &\approx P\left(Z < \frac{30/100 - 1/5}{1/30}\right) \quad \text{(by the central limit theorem)} \\ &= P(Z < 3) \\ &= 0.9987 \quad \text{(from the table of normal probabilities)} \end{aligned}$$

13. (**More Central Limit Theorem**)
Let X_j be the IQ of a randomly selected person. We are given $E(X_j) = 100$ and $\sigma_{X_j} = 15$.
Let $\overline{X}$ be the average of the IQ's of 100 randomly selected people. Then we know

$$E(\overline{X}) = 100 \quad \text{and} \quad \sigma_{\overline{X}} = 15/\sqrt{100} = 1.5.$$

The problem asks for $P(\overline{X} > 115)$. Standardizing we get $P(\overline{X} > 115) \approx P(Z > 10)$. $\boxed{\text{This is effectively 0.}}$