Practice Exam 1: Long List
18.05, Spring 2014

1 Counting and Probability

1. A full house in poker is a hand where three cards share one rank and two cards share another rank. How many ways are there to get a full-house? What is the probability of getting a full-house?

2. There are 3 arrangements of the word DAD, namely DAD, ADD, and DDA. How many arrangements are there of the word PROBABILITY?

3. **(a)** How many ways can you arrange the letters in the word STATISTICS? (e.g. SSSTTTIIAC counts a one arrangement.)

(b) If all arrangements are equally likely, what is the probabilitiy the two 'i's are next to each other.

4. There are six men and seven women in a ballroom dancing class. If four men and four women are chosen and paired off, how many pairings are possible?

5. Suppose you pick two cards from a deck of 52 playing cards. What is the probability that they are both queens?

6. Suppose that there are ten students in a classroom. What is the probability that no two of them have a birthday in the same month?

7. 20 politicians are having a tea party, 6 Democrats and 14 Republicans. To prepare, they need to choose:

3 people to set the table, 2 people to boil the water, 6 people to make the scones.

Each person can only do 1 task. (Note that this doesn't add up to 20. The rest of the people don't help.)

(a) In how many different ways can they choose which people perform these tasks?

(b) Suppose that the Democrats all hate tea. If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

(c) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 9 Republicans and 1 Democrat?

8. Let A and B be two events. Suppose the probability that neither A or B occurs is 2/3. What is the probability that one or both occur?

9. Let C and D be two events with $P(C) = 0.25$, $P(D) = 0.45$, and $P(C \cap D) = 0.1$. What is $P(C^c \cap D)$?

10. You roll a four-sided die 3 times. For this problem we'll use the sample space with 64 equally likely outcomes.

(a) Write down this sample space in set notation.

(b) List all the outcomes in each of the following events.

(i) A = 'Exactly 2 of the 3 rolls are fours'

(ii) B = 'At least 2 of the 3 rolls are fours'

(iii) C = 'Exactly 1 of the second and third rolls is a 4'

(iv) $A \cap C$

11. Suppose we have 8 teams labeled T_1, ..., T_8. Suppose they are ordered by placing their names in a hat and drawing the names out one at a time.

(a) How many ways can it happen that all the odd numbered teams are in the odd numbered slots and all the even numbered teams are in the even numbered slots?

(b) What is the probability of this happening?

12. (Taken from the book by Dekking et. al. problem 4.9) The space shuttle has 6 O-rings (these were involved in the *Challenger* disaster). When launched at 81° F, each O-ring has a probability of failure of 0.0137 (independent of whether other O-rings fail).
(a) What is the probability that during 23 launches no O-ring will fail, but that at least one O-ring will fail during the 24th launch of a space shuttle?

(b) What is the probability that no O-ring fails during 24 launches?

2 Conditional Probability and Bayes' Theorem

13. More cards! Suppose you want to divide a 52 card deck into four hands with 13 cards each. What is the probability that each hand has a king?

14. Suppose you are taking a multiple-choice test with c choices for each question. In answering a question on this test, the probability that you know the answer is p. If you don't know the answer, you choose one at random. What is the probability that you knew the answer to a question, given that you answered it correctly?

15. Corrupted by their power, the judges running the popular game show *America's Next Top Mathematician* have been taking bribes from many of the contestants. Each episode, a given contestant is either allowed to stay on the show or is kicked off.

If the contestant has been bribing the judges she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3.

Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds, i.e., if a contestant bribes them in the first round, she bribes them in the second round too (and vice versa).

(a) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

(b) If you pick a random contestant, what is the probability that she is allowed to stay during both of the first two episodes?

(c) If you pick random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

16. Consider the Monty Hall problem. Let's label the door with the car behind it a and the other two doors b and c. In the game the contestant chooses a door and then Monty chooses a door, so we can label each outcome as 'contestant followed by Monty', e.g ab means the contestant chose a and Monty chose b.

(a) Make a 3×3 probability table showing probabilities for all possible outcomes.

(b) Make a probability tree showing all possible outcomes.

(c) Suppose the contestant's strategy is to switch. List all the outcomes in the event 'the contestant wins a car'. What is the probability the contestant wins?

(d) Redo part (c) with the strategy of not switching.

17. Two dice are rolled.
A = 'sum of two dice equals 3'
B = 'sum of two dice equals 7'
C = 'at least one of the dice shows a 1'

(a) What is $P(A|C)$?

(b) What is $P(B|C)$?

(c) Are A and C independent? What about B and C?

18. There is a screening test for prostate cancer that looks at the level of PSA (prostate-specific antigen) in the blood. There are a number of reasons besides prostate cancer that a man can have elevated PSA levels. In addition, many types of prostate cancer develop so slowly that that they are never a problem. Unfortunately there is currently no test to distinguish the different types and using the test is controversial because it is hard to quantify the accuracy rates and the harm done by false positives.

For this problem we'll call a positive test a true positive if it catches a dangerous type of prostate cancer. We'll assume the following numbers:

Rate of prostate cancer among men over 50 = 0.0005
True positive rate for the test = 0.9
False positive rate for the test = 0.01

Let T be the event a man has a positive test and let D be the event a man has a dangerous type of the disease. Find $P(D|T)$ and $P(D|T^c)$.

19. A multiple choice exam has 4 choices for each question. A student has studied enough so that the probability they will know the answer to a question is 0.5, the probability that they will be able to eliminate one choice is 0.25, otherwise all 4 choices seem equally plausible. If they know the answer they will get the question right. If not they have to guess from the 3 or 4 choices.

As the teacher you want the test to measure what the student knows. If the student answers a question correctly what's the probability they knew the answer?

20. Suppose you have an urn containing 7 red and 3 blue balls. You draw three balls at random. On each draw, if the ball is red you set it aside and if the ball is blue you put it back in the urn. What is the probability that the third draw is blue?

(If you get a blue ball it counts as a draw even though you put it back in the urn.)

21. Some games, like tennis or ping pong, reach a state called *deuce*. This means that the score is tied and a player wins the game when they get *two* points ahead of the other player. Suppose the probability that you win a point is p and this is true independently for all points. If the game is at deuce what is the probability you win the game?

This is a tricky problem, but amusing if you like puzzles.

3 Independence

22. Suppose that $P(A) = 0.4, P(B) = 0.3$ and $P((A \cup B)^C) = 0.42$. Are A and B independent?

23. Suppose now that events A, B and C are *mutually independent* with

$$P(A) = 0.3, \quad P(B) = 0.4, \quad P(C) = 0.5.$$

Compute the following: (Hint: Use a Venn diagram)
(i) $P(A \cap B \cap C^c)$ (ii) $P(A \cap B^c \cap C)$ (iii) $P(A^c \cap B \cap C)$

24. You roll a twenty-sided die. Determine whether the following pairs of events are independent.

(a) 'You roll an even number' and 'You roll a number less than or equal to 10'.

(b) 'You roll an even number' and 'You roll a prime number'.

25. Suppose A and B are events with $0 < P(A) < 1$ and $0 < P(B) < 1$.
(a) If A and B are disjoint can they be independent?

(b) If A and B are independent can they be disjoint?

(c) If $A \subset B$ can they be independent?

4 Expectation and Variance

26. Directly from the definitions of expected value and variance, compute $E(X)$ and $\text{Var}(X)$ when X has probability mass function given by the following table:

X	-2	-1	0	1	2
p(X)	1/15	2/15	3/15	4/15	5/15

27. Suppose that X takes values between 0 and 1 and has probability density function $2x$. Compute $\text{Var}(X)$ and $\text{Var}(X^2)$.

28. The random variable X takes values -1, 0, 1 with probabilities 1/8, 2/8, 5/8 respectively.

(a) Compute $E(X)$.

(b) Give the pmf of $Y = X^2$ and use it to compute $E(Y)$.

(c) Instead, compute $E(X^2)$ directly from an extended table.

(d) Compute $\mathrm{Var}(X)$.

29. Suppose X is a random variable with $E(X) = 5$ and $\mathrm{Var}(X) = 2$. What is $E(X^2)$?

30. Compute the expectation and variance of a Bernoulli(p) random variable.

31. Suppose 100 people all toss a hat into a box and then proceed to randomly pick out a hat. What is the expected number of people to get their own hat back.

Hint: express the number of people who get their own hat as a sum of random variables whose expected value is easy to compute.

32. Suppose I play a gambling game where I either win or lose k dollars. Suppose further that the chance of winning is $p = .5$.

I employ the following strategy to try to guarantee that I win some money.
I bet \$1; if I lose, I double my bet to \$2, if I lose I double my bet again. I continue until I win. Eventually I'm sure to win a bet and net \$1 (run through the first few rounds and you'll see why this is the net).
If this really worked casinos would be out of business. Our goal in this problem is to understand the flaw in the strategy.

(a) Let X be the amount of money bet on the last game (the one I win). X takes values 1, 2, 4, 8, ... Determine the probability mass function for X. That is, find $p(2^k)$, where k is in $\{0, 1, 2, \ldots\}$.

(b) Compute $E(X)$.

(c) Use your answer in part (b) to explain why the stategy is a bad one.

5 Probability Mass Functions, Probability Density Functions and Cumulative Distribution Functions

33. Suppose that $X \sim \mathrm{Bin}(n, 0.5)$. Find the probability mass function of $Y = 2X$.

34. **(a)** Suppose that X is uniform on $[0, 1]$. Compute the pdf and cdf of X.

(b) If $Y = 2X + 5$, compute the pdf and cdf of Y.

35. **(a)** Suppose that X has probability density function $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Compute the cdf, $F_X(x)$.

(b) If $Y = X^2$, compute the pdf and cdf of Y.

36. Suppose that X is a random variable that takes on values 0, 2 and 3 with probabilities 0.3, 0.1, 0.6 respectively. Let $Y = 3(X - 1)^2$.

(a) What is the expectation of X?

(b) What is the variance of X?

(c) What is the expection of Y?

(d) Let $F_Y(t)$ be the cumulative density function of Y. What is $F_Y(7)$?

37. Suppose you roll a fair 6-sided die 100 times (independently), and you get \$3 every time you roll a 6. Let X_1 be the number of dollars you win on rolls 1 through 25.

Let X_2 be the number of dollars you win on rolls 26 through 50.

Let X_3 be the number of dollars you win on rolls 51 through 75.

Let X_4 be the number of dollars you win on rolls 76 throught 100.

Let $X = X_1 + X_2 + X_3 + X_4$ be the total number of dollars you win over all 100 rolls.

(a) What is the probability mass function of X?

(b) What is the expectation and variance of X?

(c) Let $Y = 4X_1$. (So instead of rolling 100 times, you just roll 25 times and multiply your winnings by 4.) (i) What are the expectation and variance of Y?
(ii) How do the expectation and variance of Y compare to those of X? (I.e., are they bigger, smaller, or equal?) Explain (briefly) why this makes sense.

38. Let R be the rate at which customers are served in a queue. Suppose that R is exponential with pdf $f(r) = 2e^{-2r}$ on $[0, \infty)$.

Find the pdf of the waiting time per customer $T = 1/R$.

39. A continuous random variable X has PDF $f(x) = x + ax^2$ on [0,1]
Find a, the CDF and $P(.5 < X < 1)$.

40. (**PMF of a sum**) Suppose X and Y are independent and $X \sim$ Bernoulli(1/2) and $Y \sim$ Bernoulli(1/3). Determine the pmf of $X + Y$

41. Let X be a discrete random variable with pmf p given by:

x	-2	-1	0	1	2
$p(x)$	1/15	2/15	3/15	4/15	5/15

(a) Let $Y = X^2$. Find the pmf of Y.

(b) Find the value the cdf of X at -1/2, 3/4, 7/8, 1, 1.5, 5.

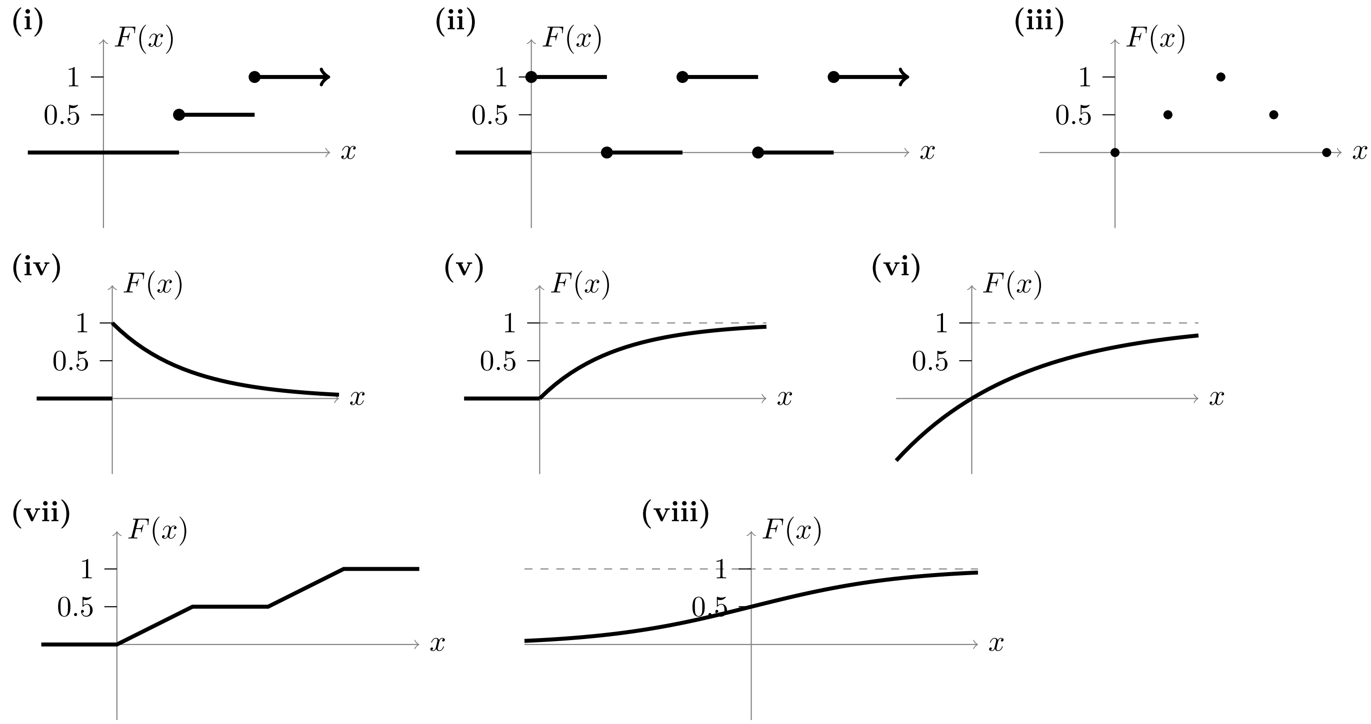
(c) Find the value the cdf of Y at -1/2, 3/4, 7/8, 1, 1.5, 5.

42. Suppose that the cdf of X is given by:

$$F(a) = \begin{cases} 0 & \text{for } a < 0 \\ \frac{1}{5} & \text{for } 0 \le a < 2 \\ \frac{2}{5} & \text{for } 2 \le a < 4 \\ 1 & \text{for } a \ge 4. \end{cases}$$

Determine the pmf of X.

43. For each of the following say whether it can be the graph of a cdf. If it can be, say whether the variable is discrete or continuous.



44. Suppose X has range [0,1] and has cdf

$$F(x) = x^2 \quad \text{for } 0 \le x \le 1.$$

Compute $P(\frac{1}{2} < X < \frac{3}{4})$.

45. Let X be a random variable with range $[0, 1]$ and cdf

$$F(X) = 2x^2 - x^4 \quad \text{for } 0 \le x \le 1.$$

(a) Compute $P(\frac{1}{4} \le X \le \frac{3}{4})$.

(b) What is the pdf of X?

6 Distributions with Names

Exponential Distribution

46. Suppose that buses arrive are scheduled to arrive at a bus stop at noon but are always X minutes late, where X is an exponential random variable with probability density function $f_X(x) = \lambda e^{-\lambda x}$. Suppose that you arrive at the bus stop precisely at noon.

(a) Compute the probability that you have to wait for more than five minutes for the bus to arrive.

(b) Suppose that you have already waiting for 10 minutes. Compute the probability that you have to wait an additional five minutes or more.

47. Normal Distribution: Throughout these problems, let ϕ and Φ be the pdf and cdf, respectively, of the standard normal distribution Suppose Z is a standard normal random variable and let $X = 3Z + 1$.

(a) Express $P(X \leq x)$ in terms of Φ

(b) Differentiate the expression from (a) with respect to x to get the pdf of X, $f(x)$. Remember that $\Phi'(z) = \phi(z)$ and don't forget the chain rule

(c) Find $P(-1 \leq X \leq 1)$

(d) Recall that the probability that Z is within one standard deviation of its mean is approximately 68%. What is the probability that X is within one standard deviation of its mean?

48. Transforming Normal Distributions
Suppose $Z \sim$ N(0,1) and $Y = \mathrm{e}^Z$.

(a) Find the cdf $F_Y(a)$ and pdf $f_Y(y)$ for Y. (For the CDF, the best you can do is write it in terms of Φ the standard normal cdf.)

(b) We don't have a formula for $\Phi(z)$ so we don't have a formula for quantiles. So we have to write quantiles *in terms* of Φ^{-1}.
(i) Write the 0.33 quantile of Z in terms of Φ^{-1}
(ii) Write the 0.9 quantile of Y in terms of Φ^{-1}.
(iii) Find the median of Y.

49. (**Random variables derived from normal r.v.**)
Let X_1, X_2, $\ldots X_n$ be i.i.d. $\mathrm{N}(0, 1)$ random variables.
Let $Y_n = X_1^2 + \ldots + X_n^2$.

(a) Use the formula $\mathrm{Var}(X_j) = E(X_j^2) - E(X_j)^2$ to show $E(X_j^2) = 1$.

(b) Set up an integral in x for computing $E(X_j^4)$.

For 3 extra credit points, use integration by parts show $E(X_j^4) = 3$.
(If you don't do this, you can still use the result in part c.)

(c) Deduce from parts (a) and (b) that $\mathrm{Var}(X_j^2) = 2$.

(d) Use the Central Limit Theorem to approximate $P(Y_{100} > 110)$.

50. More Transforming Normal Distributions
(a) Suppose Z is a standard normal random variable and let $Y = aZ + b$, where $a > 0$ and b are constants.

Show $Y \sim \mathrm{N}(b, a)$.

(b) Suppose $Y \sim \mathrm{N}(\mu, \sigma)$. Show $\dfrac{Y - \mu}{\sigma}$ follows a standard normal distribution.

51. (**Sums of normal random variables**)
Let X be independent random variables where $X \sim N(2, 5)$ and $Y \sim N(5, 9)$ (we use the notation $N(\mu, \sigma^2)$). Let $W = 3X - 2Y + 1$.

(a) Compute $E(W)$ and $\mathrm{Var}(W)$.

(b) It is known that the sum of independent normal distributions is normal. Compute

$P(W \leq 6)$.

52. Let $X \sim \mathrm{U}(a, b)$. Compute $E(X)$ and $\mathrm{Var}(X)$.

53. In $n + m$ independent Bernoulli(p) trials, let S_n be the number of successes in the first n trials and T_m the number of successes in the last m trials.

(a) What is the distribution of S_n? Why?

(b) What is the distribution of T_m? Why?

(c) What is the distribution of $S_n + T_m$? Why?

(d) Are S_n and T_m independent? Why?

54. Compute the median for the exponential distribution with parameter λ. (The exponential distribution is defined on page 62 of the text.)

55. Pareto and the 80-20 rule.

Pareto was an economist who used the Pareto distribution to model the wealth in a society. For a fixed baseline m, the Pareto density with parameter α is

$$f(x) = \frac{\alpha\, m^\alpha}{x^{\alpha+1}} \quad \text{for } x \geq m.$$

Assume X is a random variable that follows such a distribution.

(a) Compute $P(X > a)$ (you may assume $a \geq m$).

(b) Pareto's principle is often paraphrased as the 80-20 rule. That is, 80% of the wealth is owned by 20% of the people. The rule is only exact for a Pareto distribution with $\alpha = \log(5)/\log(4) = 1.16$.

Suppose $\alpha = m = 1$. Compute the 0.80 quantile for the Pareto distribution.

In general, many phenomena follow the power law described by $f(x)$. You can look up 'Pareto principle' in Wikipedia to read more about this.

7 Joint Probability, Covariance, Correlation

56. (**Another Arithmetic Puzzle**) Let X and Y be two independent Bernoulli(.5) random variables. Define S and T by:

$$S = X + Y \quad \text{and} \quad T = X - Y.$$

(a) Find the joint and marginal pmf's for S and T.

(b) Are S and T independent.

57. Data is taken on the height and shoe size of a sample of MIT students. Height is coded by 3 values: 1 (short), 2 (average), 3 (tall) and shoe size is coded by 3 values 1 (small), 2 (average), 3 (large). The joint counts are given in the following table.

Shoe \ Height	1	2	3
1	234	225	84
2	180	453	161
3	39	192	157

Let X be the coded shoe size and Y the height of a random person in the sample.

(a) Find the joint and marginal pmf of X and Y.

(b) Are X and Y independent?

58. Let X and Y be two continuous random variables with joint pdf

$$f(x, y) = cx^2 y(1 + y) \quad \text{for } 0 \le x \le 3 \text{ and } 0 \le y \le 3,$$

and $f(x, y) = 0$ otherwise.

(a) Find the value of c.

(b) Find the probability $P(1 \le X \le 2,\ 0 \le Y \le 1)$.

(c) Determine the joint cdf of X and Y for a and b between 0 and 3.

(d) Find marginal cdf $F_X(a)$ for a between 0 and 3.

(e) Find the marginal pdf $f_X(x)$ directly from $f(x, y)$ and check that it is the derivative of $F_X(x)$.

(f) Are X and Y independent?

59. Let X and Y be two random variables and let r, s, t, and u be real numbers.
(a) Show that $\text{Cov}(X + s, Y + u) = \text{Cov}(X, Y)$.

(b) Show that $\text{Cov}(rX, tY) = rt\text{Cov}(X, Y)$.

(c) Show that $\text{Cov}(rX + s, tY + u) = rt\text{Cov}(X, Y)$.

60. Derive the formula for the covariance: $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$.

61. (**Arithmetic Puzzle**) The joint and marginal pmf's of X and Y are partly given in the following table.

$X \backslash Y$	1	2	3	
1	1/6	0	...	1/3
2	...	1/4	...	1/3
3	...	...	1/4	...
	1/6	1/3	...	1

(a) Complete the table.

(b) Are X and Y independent?

62. (**Simple Joint Probability**) Let X and Y each have range {1,2,3,4}. The following formula gives their joint pmf

$$P(X = i, Y = j) = \frac{i + j}{80}$$

Compute each of the following:

(a) $P(X = Y)$.

(b) $P(XY = 6)$.

(c) $P(1 \le X \le 2,\ 2 < Y \le 4)$.

63. Toss a fair coin 3 times. Let X = the number of heads on the first toss, Y the total number of heads on the last two tosses, and F the number of heads on the first two tosses.

(a) Give the joint probability table for X and Y. Compute $\text{Cov}(X, Y)$.

(b) Give the joint probability table for X and F. Compute $\text{Cov}(X, F)$.

64. Covariance and Independence
Let X be a random variable that takes values -2, -1, 0, 1, 2; each with probability 1/5. Let $Y = X^2$.

(a) Fill out the following table giving the joint frequency function for X and Y. Be sure to include the marginal probabilities.

X / Y	-2	-1	0	1	2	total
0						
1						
4						
total						

(b) Find $E(X)$ and $E(Y)$.

(c) Show X and Y are not independent.

(d) Show $\text{Cov}(X, Y) = 0$.
This is an example of uncorrelated but non-independent random variables. The reason this can happen is that correlation only measures the linear dependence between the two variables. In this case, X and Y are not at all linearly related.

65. Continuous Joint Distributions
Suppose X and Y are continuous random variables with joint density function $f(x, y) = x+y$ on the unit square $[0, 1] \times [0, 1]$.

(a) Let $F(x, y)$ be the joint CDF. Compute $F(1, 1)$. Compute $F(x, y)$.

(b) Compute the marginal densities for X and Y.

(c) Are X and Y independent?

(d) Compute $E(X)$, (Y), $E(X^2 + Y^2)$, $\text{Cov}(X, Y)$.

8 Law of Large Numbers, Central Limit Theorem

66. Suppose $X_1, \ldots, X_{100}$ are i.i.d. with mean 1/5 and variance 1/9. Use the central limit theorem to estimate $P(\sum X_i < 30)$.

67. All or None
You have \$100 and, never mind why, you must convert it to \$1000. Anything less is no good. Your only way to make money is to gamble for it. Your chance of winning one bet is p.

Here are two extreme strategies:

Maximum strategy: bet as much as you can each time. To be smart, if you have less than \$500 you bet it all. If you have more, you bet enough to get to \$1000.

Minimum strategy: bet \$1 each time.

If $p < .5$ (the odds are against you) which is the better strategy?
What about $p > .5$ or $p = .5$?

68. (**Central Limit Theorem**) Let $X_1, X_2, \ldots, X_{81}$ be i.i.d., each with expected value $\mu = E(X_i) = 5$, and variance $\sigma^2 = \text{Var}(X_i) = 4$. Approximate $P(X_1 + X_2 + \cdots X_{81} > 369)$, using the central limit theorem.

69. (**Binomial $\approx$ normal**) Let $X \sim$ binomial(100,1/3).
An 'exact' computation in R gives $P(X \le 30) = 0.2765539$. Use the central limit theorem to give an approximation of $P(X \le 30)$

70. (**More Central Limit Theorem**)
The average IQ in a population is 100 with standard deviation 15 (by definition, IQ is normalized so this is the case). What is the probability that a randomly selected group of 100 people has an average IQ above 115?

9 R Problems

R will not be on the exam. However, these problems will help you understand the concepts we've been studying.

71. R simulation
Consider X_1, X_2, ... all independent and with distribution $\text{N}(0, 1)$. Let $\overline{X}_m$ be the average of X_1, ... X_n.

(a) Give $E(\overline{X}_n)$ and $\sigma_{\overline{X}_n}$ exactly.

(b) Use a R simulation to estimate $E(\overline{X}_n)$ and $\text{Var}(\overline{X}_n)$ for $n = 1, 9, 100$. (You should use the `rnorm` function to simulate 1000 samples of each X_j.)

72. R Exercise
Let X_1, X_2, X_3, X_4, X_5 be independent $U(0, 1)$ random variables.
Let $X = X_1 + X_2 + X_3$ and $Y = X_3 + X_4 + X_5$.

Use the `runif()` function to simulate 1000 trials of each of these variables. Use these to estimate $\text{Cov}(X, Y)$.

Extra Credit
Compute this covariance exactly.

18.05 Introduction to Probability and Statistics
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