Exam 1 Practice Exam 1: Long List –solutions, 18.05, Spring 2014

1 Counting and Probability

1. We build a full-house in stages and count the number of ways to make each stage:

Stage 1. Choose the rank of the pair: $\binom{13}{1}$.

Stage 2. Choose the pair from that rank, i.e. pick 2 of 4 cards: $\binom{4}{2}$.

Stage 3. Choose the rank of the triple (from the remaining 12 ranks): $\binom{12}{1}$.

Stage 4. Choose the triple from that rank: $\binom{4}{3}$.

Number of ways to get a full-house: $\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3}$

Number of ways to pick any 5 cards out of 52: $\binom{52}{5}$

Probability of a full house: $\dfrac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{14}{3}}{\binom{52}{5}} \approx 0.00144$

2. Sort the letters: A BB II L O P R T Y. There are 11 letters in all. We build arrangements by starting with 11 'slots' and placing the letters in these slots, e.g

$$\underline{A}\ \underline{B}\ \underline{I}\ \underline{B}\ \underline{I}\ \underline{L}\ \underline{O}\ \underline{P}\ \underline{R}\ \underline{T}\ \underline{Y}$$

Create an arrangement in stages and count the number of possibilities at each stage:

Stage 1: Choose one of the 11 slots to put the A: $\binom{11}{1}$

Stage 2: Choose two of the remaining 10 slots to put the B's: $\binom{10}{2}$

Stage 3: Choose two of the remaining 8 slots to put the B's: $\binom{8}{2}$

Stage 4: Choose one of the remaining 6 slots to put the L: $\binom{6}{1}$

Stage 5: Choose one of the remaining 5 slots to put the O: $\binom{5}{1}$

Stage 6: Choose one of the remaining 4 slots to put the P: $\binom{4}{1}$

Stage 7: Choose one of the remaining 3 slots to put the R: $\binom{3}{1}$

Stage 8: Choose one of the remaining 2 slots to put the T: $\binom{2}{1}$

Stage 9: Use the last slot for the Y: $\binom{1}{1}$

Number of arrangements:

$$\binom{11}{1}\binom{10}{2}\binom{8}{2}\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = 11\cdot\frac{10\cdot 9}{2}\cdot\frac{8\cdot 7}{2}\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 9979200$$

Note: choosing 11 out of 1 is so simple we could have immediately written 11 instead of belaboring the issue by writing $\binom{11}{1}$. We wrote it this way to show one systematic way to think about problems like this.

3. **(a)** Create an arrangement in stages and count the number of possibilities at each stage:
Stage 1: Choose three of the 10 slots to put the S's: $\binom{10}{3}$
Stage 2: Choose three of the remaining 7 slots to put the T's: $\binom{7}{3}$
Stage 3: Choose two of the remaining 4 slots to put the I's: $\binom{4}{2}$
Stage 4: Choose one of the remaining 2 slots to put the A: $\binom{2}{1}$
Stage 5: Use the last slot for the C: $\binom{1}{1}$
Number of arrangements:

$$\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 50500.$$

(b) The are $\binom{10}{2} = 45$ equally likely ways to place the two I's.

There are 9 ways to place them next to each other, i.e. in slots 1 and 2, slots 2 and 3, ..., slots 9 and 10.

So the probability the I's are adjacent is $9/45 = 0.2$.

4. Build the pairings in stages and count the ways to build each stage:
Stage 1: Choose the 4 men: $\binom{6}{4}$.
Stage 2: Choose the 4 women: $\binom{7}{4}$
We need to be careful because we don't want to build the same 4 couples in multiple ways.
Line up the 4 men M_1, M_2, M_3, M_4
Stage 3: Choose a partner from the 4 women for M_1: 4.
Stage 4: Choose a partner from the remaining 3 women for M_2: 3
Stage 5: Choose a partner from the remaining 2 women for M_3: 2
Stage 6: Pair the last women with M_4: 1

Number of possible pairings: $\binom{6}{4}\binom{7}{4}4!$.

Note: we could have done stages 3-6 in on go as: Stages 3-6: Arrange the 4 women opposite the 4 men: 4! ways.

5. Using choices (order doesn't matter):
Number of ways to pick 2 queens: $\binom{4}{2}$. Number of ways to pick 2 cards: $\binom{52}{2}$.

All choices of 2 cards are equally likely. So, probability of 2 queens $= \dfrac{\binom{4}{2}}{\binom{52}{2}}$

Using permutations (order matters):
Number of ways to pick the first queen: 4. No. of ways to pick the second queen: 3.
Number of ways to pick the first card: 52. No. of ways to pick the second card: 51.
All arrangements of 2 cards are equally likely. So, probability of 2 queens: $\frac{4\cdot 3}{52\cdot 51}$.

6. We assume each month is equally likely to be a student's birthday month.
Number of ways ten students can have birthdays in 10 different months:

$$12\cdot 11\cdot 10\ldots\cdot 3 \;=\; \frac{12!}{2!}$$

Number of ways 10 students can have birthday months: 12^{10}.

Probability no two share a birthday month: $\dfrac{12!}{2!\,12^{10}} \;=\; 0.00387$

7. **(a)** There are $\binom{20}{3}$ ways to choose the 3 people to set the table, then $\binom{17}{2}$ ways to choose the 2 people to boil water, and $\binom{15}{6}$ ways to choose the people to make scones. So the total number of ways to choose people for these tasks is

$$\boxed{\binom{20}{3}\binom{17}{2}\binom{15}{6} \;=\; \frac{20!}{3!\,17!}\cdot\frac{17!}{2!\,15!}\cdot\frac{15!}{6!\,9!} \;=\; \frac{20!}{3!\,2!\,6!\,9!} \;=\; 775975200.}$$

(b) The number of ways to choose 10 of the 20 people is $\binom{20}{10}$ The number of ways to choose 10 people from the 14 Republicans is $\binom{14}{10}$. So the probability that you only choose 10 Republicans is

$$\frac{\binom{14}{10}}{\binom{20}{10}} = \frac{\frac{14!}{10!\,4!}}{\frac{20!}{10!\,10!}} \approx 0.00542$$

Alternatively, you could choose the 10 people in sequence and say that there is a 14/20 probability that the first person is a Republican, then a 13/19 probability that the second one is, a 12/18 probability that third one is, etc. This gives a probability of

$$\frac{14}{20}\cdot\frac{13}{19}\cdot\frac{12}{18}\cdot\frac{11}{17}\cdot\frac{10}{16}\cdot\frac{9}{15}\cdot\frac{8}{14}\cdot\frac{7}{13}\cdot\frac{6}{12}\cdot\frac{5}{11}.$$

(You can check that this is the same as the other answer given above.)

(c) You can choose 1 Democrat in $\binom{6}{1} = 6$ ways, and you can choose 9 Republicans in $\binom{14}{9}$ ways, so the probability equals

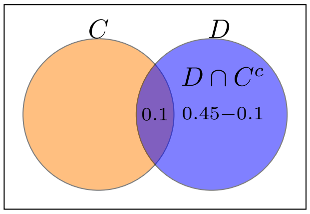
$$\frac{6\cdot\binom{14}{9}}{\binom{20}{10}} = \frac{6\cdot\frac{14!}{9!\,5!}}{\frac{20!}{10!\,10!}} = \frac{6\cdot 14!\,10!\,10!}{9!\,5!\,20!}.$$

8. We are given $P(A^c \cap B^c) = 2/3$ and asked to find $P(A \cup B)$.

$A^c \cap B^c = (A \cup B)^c \Rightarrow P(A \cup B) = 1 - P(A^c \cap B^c) = \boxed{1/3.}$

9. D is the disjoint union of $D \cap C$ and $D \cap C^c$.

So, $P(D \cap C) + P(D \cap C^c) = P(D)$
$\Rightarrow P(D \cap C^c) = P(D) - P(D \cap C) = 0.45 - 0.1 = \boxed{0.35.}$
(We never use $P(C) = 0.25$.)



10. **(a)** Writing all 64 possibilities is too tedius. Here's a more compact representation

$$\{(i, j, k) \,|\, i, j, k \text{ are integers from 1 to 4}\}$$

(b) (i) Here we'll just list all 9 possibilities
{(4,4,1), (4,4,2), (4,4,3), (4,1,4), (4,2,4), (4,3,4), (1,4,4), (2,4,4), (3,4,4)}

(ii) This is the same as (i) with the addition of (4,4,4).
{ (4,4,1), (4,4,2), (4,4,3), (4,1,4), (4,2,4), (4,3,4), (1,4,4), (2,4,4), (3,4,4), (4,4,4)}

(iii) This is list is a little longer. If we're systematic about it we can still just write it out.

{(1,4,1), (2,4,1), (3,4,1), (4,4,1),
(1,4,2), (2,4,2), (3,4,2), (4,4,2),
(1,4,3), (2,4,3), (3,4,3), (4,4,3),
(1,1,4), (2,1,4), (3,1,4), (4,1,4),
(1,2,4), (2,2,4), (3,2,4), (4,2,4),
(1,3,4), (2,3,4), (3,3,4), (4,3,4)}

(iv) {(4,4,1), (4,4,2), (4,4,3), (4,1,4), (4,2,4), (4,3,4)}

11. **(a)** Slots 1, 3, 5, 7 are filled by T_1, T_3, T_5, T_7 in any order: 4! ways.
Slots 2, 4, 6, 8 are filled by T_2, T_4, T_6, T_8 in any order: 4! ways.
answer: $4! \cdot 4! = 576$.

(b) There are 8! ways to fill the 8 slots in any way.
Since each outcome is equally likely the probabilitiy is $\dfrac{4! \cdot 4!}{8!} = \dfrac{576}{40320} = 0.143 = 1.43\%$.

12. **(a)** $P(\text{one O-ring fails}) = p = 0.0137$.
Failure of 1 ring follows a Bernoulli(p) distribution.
Let X be the number of O-ring failures in a launch. We assume O-rings fail independently.
There are 6 O-rings per launch so $X \sim \text{Binomial}(6, p)$.
For convenience call $P(X = 0) = (1 - p)^6 := q$.
Let A be the event asked about: no failures in first 23 launces and at least one failure in the 24th.
$P(A) = q^{23}(1 - q) = \boxed{0.0118 = 1.18\%.}$

(b) $P(\text{no failures in 24 launches}) = q^{24} = 0.137 = 13.7\%$.

2 Conditional Probability and Bayes' Theorem

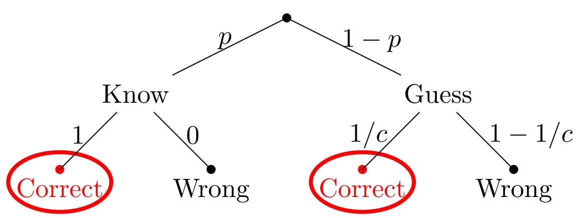
13. Let H_i be the event that the i^{th} hand has one king. We have the conditional probabilities

$$P(H_1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}; \quad P(H_2|H_1) = \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}}; \quad P(H_3|H_1 \cap H_2) = \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}}$$

$P(H_4|H_1 \cap H_2 \cap H_3) = 1$

$$\begin{aligned} P(H_1 \cap H_2 \cap H_3 \cap H_4) &= P(H_4|H_1 \cap H_2 \cap H_3)\, P(H_3|H_1 \cap H_2)\, P(H_2|H_1)\, P(H_1) \\ &= \frac{\binom{2}{1}\binom{24}{12}\binom{3}{1}\binom{36}{12}\binom{4}{1}\binom{48}{12}}{\binom{26}{13}\binom{39}{13}\binom{52}{13}}. \end{aligned}$$

14. The following tree shows the setting



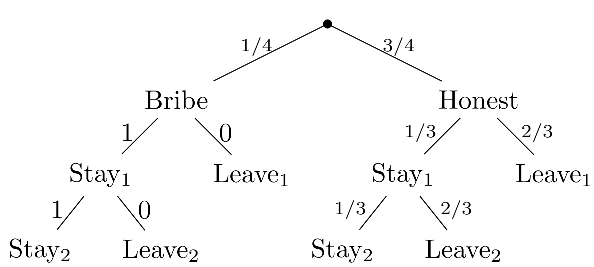
Let C be the event that you answer the question correctly. Let K be the event that you actually know the answer. The left circled node shows $P(K \cap C) = p$. Both circled nodes together show $P(C) = p + (1-p)/c$. So,

$$P(K|C) = \frac{P(K \cap C)}{P(C)} = \frac{p}{p + (1-p)/c}$$

Or we could use the algebraic form of Bayes' theorem and the law of total probability: Let G stand for the event that you're guessing. Then we have,
$P(C|K) = 1$, $P(K) = p$, $P(C) = P(C|K)P(K) + P(C|G)P(G) = p + (1-p)/c$. So,

$$P(K|C) = \frac{P(C|K)P(K)}{P(C)} = \frac{p}{p + (1-p)/c}$$

15. The following tree shows the setting. Stay$_1$ means the contestant was allowed to stay during the first episode and stay$_2$ means the they were allowed to stay during the second.



Let's name the relevant events:
B = the contestant is bribing the judges
H = the contestant is honest (not bribing the judges)
S_1 = the contestant was allowed to stay during the first episode
S_2 = the contestant was allowed to stay during the second episode
L_1 = the contestant was asked to leave during the first episode
L_2 = the contestant was asked to leave during the second episode

(a) We first compute $P(S_1)$ using the law of total probability.

$$P(S_1) = P(S_1|B)P(B) + P(S_1|H)P(H) = 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{2}.$$

We therefore have (by Bayes' rule) $P(B|S_1) = P(S_1|B)\dfrac{P(B)}{P(S_1)} = 1 \cdot \dfrac{1/4}{1/2} = \dfrac{1}{2}$.

(b) Using the tree we have the total probability of S_2 is

$$P(S_2) = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}$$

(c) We want to compute $P(L_2|S_1) = \dfrac{P(L_2 \cap S_1)}{P(S_1)}$.

From the calculation we did in part (a), $P(S_1) = 1/2$. For the numerator, we have (see the tree)

$$P(L_2 \cap S_1) = P(L_2 \cap S_1|B)P(B) + P(L_2 \cap S_1|H)P(H) = 0 \cdot \frac{1}{3} + \frac{2}{9} \cdot \frac{3}{4} = \frac{1}{6}$$

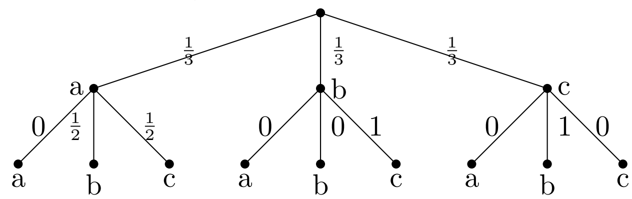
Therefore $\boxed{P(L_2|S_1) = \dfrac{1/6}{1/2} = \dfrac{1}{3}}$.

16. **(a)** and **(b)** In the tree the first row is the contestant's choice and the second row is the host's (Monty's) choice.

Viewed as a table

		Contestant		
		a	b	c
	a	0	0	0
Host	b	1/6	0	1/3
	c	1/6	1/3	0

Viewed as a tree



(b) With this strategy the contestant wins with {bc, cb}. The probability of winning is $P(bc) + P(cb) = 2/3$. (Both the tree and the table show this.)

(c) $\{ab, ac\}$, probability = 1/3.

17. Sample space =

$$\Omega = \{(1,1), (1,2), (1,3), \ldots, (6,6)\} = \{(i,j) \,|\, i,j = 1, 2, 3, 4, 5, 6\}.$$

(Each outcome is equally likely, with probability 1/36.)

$A = \{(1,2),\ (2,1)\}$,
$B = \{(1,6),\ (2,5),\ (3,4),\ (4,3),\ (5,2),\ (6,1)\}$
$C = \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (2,1),\ (3,1),\ (4,1),\ (5,1),\ (6,1)\}$

(a) $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{2/36}{11/36} = \dfrac{2}{11}$..

(a) $P(B|C) = \dfrac{P(B \cap C)}{P(C)} = \dfrac{2/36}{11/36} = \dfrac{2}{11}$..

(c) $P(A) = 2/36 \neq P(A|C)$, so they are not independent. Similarly, $P(B) = 6/36 \neq P(B|C)$, so they are not independent.

18. You should write this out in a tree! (For example, see the solution to the next problem.)

We compute all the pieces needed to apply Bayes' rule. We're given
$P(T|D) = 0.9 \Rightarrow P(T^c|D) = 0.1, \quad P(T|D^c) = 0.01 \Rightarrow P(T^c|D^c) = 0.99.$
$P(D) = 0.0005 \Rightarrow P(D^c) = 1 - P(D) = 0.9995.$

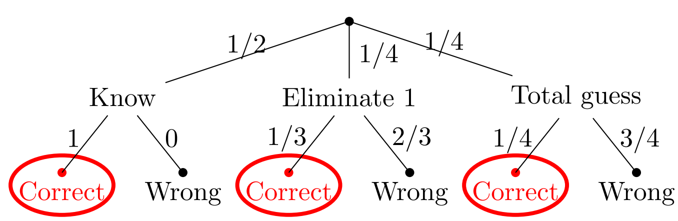
We use the law of total probability to compute $P(T)$:

$$P(T) = P(T|D)\,P(D) + P(T|D^c)\,P(D^c) = 0.9 \cdot 0.0005 + 0.01 \cdot 0.9995 = 0.010445$$

Now we can use Bayes' rule to answer the questions:

$$P(D|T) = \frac{P(T|D)\,P(D)}{P(T)} = \frac{0.9 \times 0.0005}{0.010445} = 0.043$$
$$P(D|T^c) = \frac{P(T^c|D)\,P(D)}{P(T^c)} = \frac{0.1 \times 0.0005}{0.989555} = 5.0 \times 10^{-5}$$

19. We show the probabilities in a tree:



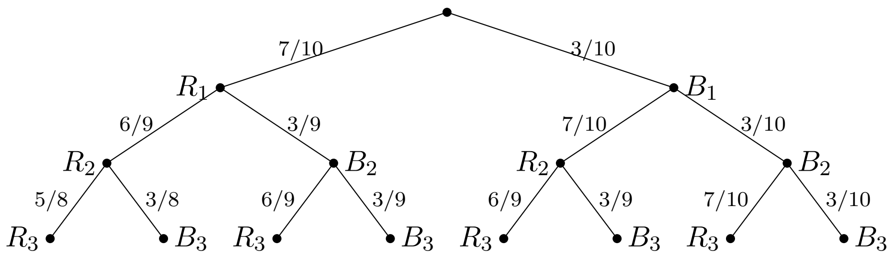
For a given problem let C be the event the student gets the problem correct and K the event the student knows the answer.

The question asks for $P(K|C)$.

We'll compute this using Bayes' rule:

$$P(K|C) = \frac{P(C|K)\,P(K)}{P(C)} = \frac{1 \cdot 1/2}{1/2 + 1/12 + 1/16} = \frac{24}{31} \approx 0.774 = 77.4\%$$

20. Here is the game tree, R_1 means red on the first draw etc.



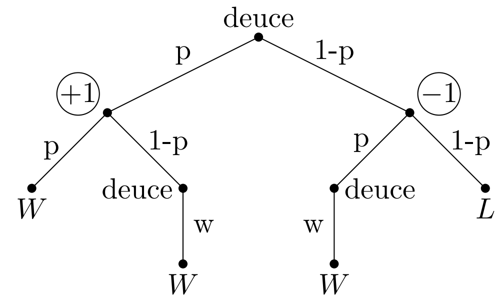
Summing the probability to all the B_3 nodes we get

$$P(B_3) = \frac{7}{10}\cdot\frac{6}{9}\cdot\frac{3}{8} + \frac{7}{10}\cdot\frac{3}{9}\cdot\frac{3}{9} + \frac{3}{10}\cdot\frac{7}{10}\cdot\frac{3}{9} + \frac{3}{10}\cdot\frac{3}{10}\cdot\frac{3}{10} = 0.350.$$

21. Let W be the event you win the game from deuce and L the event you lose. For convenience, define $w = P(W)$.

The figure shows the complete game tree through 2 points. In the third level we just abreviate by indicating the probability of winning from deuce.

The nodes marked +1 and -1, indicate whether you won or lost the first point.



Summing all the paths to W we get

$$w = P(W) = p^2 + p(1-p)w + (1-p)pw = p^2 + 2p(1-p)w \;\Rightarrow\; \boxed{w = \frac{p^2}{1-2p(1-p)}}.$$

3 Independence

22. We have $P(A\cup B) = 1 - 0.42 = 0.58$ and we know because of the inclusion-exclusion principle that

$$P(A\cup B) = P(A) + P(B) - P(A\cap B).$$

Thus,

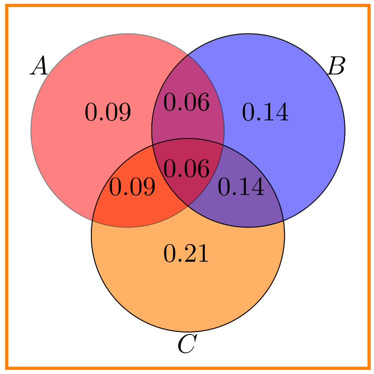
$$P(A\cap B) = P(A) + P(B) - P(A\cup B) = 0.4 + 0.3 - 0.58 = 0.12 = (0.4)(0.3) = P(A)P(B)$$

So A and B are independent.

23. By the mutual independence we have

$$P(A\cap B\cap C) = P(A)P(B)P(C) = 0.06 \qquad P(A\cap B) = P(A)P(B) = 0.12$$
$$P(A\cap C) = P(A)P(C) = 0.15 \qquad P(B\cap C) = P(B)P(C) = 0.2$$

We show this in the following Venn diagram



Note that, for instance, $P(A \cap B)$ is split into two pieces. One of the pieces is $P(A \cap B \cap C)$ which we know and the other we compute as $P(A \cap B) - P(A \cap B \cap C) = 0.12 - 0.06 = 0.06$. The other intersections are similar.

We can read off the asked for probabilities from the diagram.
(i) $P(A \cap B \cap C^c) = 0.06$
(ii) $P(A \cap B^c \cap C) = 0.09$
(iii) $P(A^c \cap B \cap C) = 0.14$.

24. E = even numbered = $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.
L = roll $\leq 10 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
B = roll is prime = $\{2, 3, 5, 7, 11, 13, 17, 19\}$ (We use B because P is not a good choice.)

(a) $P(E) = 10/20$, $P(E|L) = 5/10$. These are the same, so the events are independent.

(b) $P(E) = 10/20$. $P(E|B) = 1/8$. These are not the same so the events are not independent.

25. The answer to all three parts is 'No'. Each of these answers relies on the fact that the probabilities of A and B are strictly between 0 and 1.

To show A and B are not independent we need to show either $P(A \cap B) \neq P(A) \cdot P(B)$ or $P(A|B) \neq P(A)$.

(a) No, they cannot be independent: $A \cap B = \emptyset \Rightarrow P(A \cap B) = 0 \neq P(A) \cdot P(B)$.

(b) No, they cannot be disjoint: same reason as in part (a).

(c) No, they cannot be independent: $A \subset B \Rightarrow A \cap B = A$
$\Rightarrow P(A \cap B) = P(A) > P(A) \cdot P(B)$. The last inequality follows because $P(B) < 1$.

4 Expectation and Variance

26. *Solution:* We compute

$$E[X] = -2 \cdot \frac{1}{15} + -1 \cdot \frac{2}{15} + 0 \cdot \frac{3}{15} + 1 \cdot \frac{4}{15} + 2 \cdot \frac{5}{15} = \frac{2}{3}.$$

Thus

$$\begin{aligned}\text{Var}(X) &= E((X-\frac{2}{3})^2)\\ &= \left(-2-\frac{2}{3}\right)^2\cdot\frac{1}{15}+\left(-1-\frac{2}{3}\right)^2\cdot\frac{2}{15}+\left(0-\frac{2}{3}\right)^2\cdot\frac{3}{15}+\left(1-\frac{2}{3}\right)^2\cdot\frac{4}{15}+\left(2-\frac{2}{3}\right)^2\cdot\frac{5}{15}\\ &= \frac{14}{9}.\end{aligned}$$

27. We will make use of the formula $\text{Var}(Y) = E(Y^2) - E(Y)^2$. First we compute

$$E[X] = \int_0^1 x\cdot 2x dx = \frac{2}{3}$$

$$E[X^2] = \int_0^1 x^2\cdot 2x dx = \frac{1}{2}$$

$$E[X^4] = \int_0^1 x^4\cdot 2x dx = \frac{1}{3}.$$

Thus,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

and

$$\text{Var}(X^2) = E[X^4] - \left(E[X^2]\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

28. **(a)** We have

X values:	-1	0	1
prob:	1/8	2/8	5/8
X^2	1	0	1

So, $\boxed{E(X) = -1/8 + 5/8 = 1/2.}$

(b)

Y values:	0	1
prob:	2/8	6/8

$\Rightarrow$ $\boxed{E(Y) = 6/8 = 3/4.}$

(c) The change of variables formula just says to use the bottom row of the table in part (a): $E(X^2) = 1\cdot(1/8) + 0\cdot(2/8) + 1\cdot(5/8) = 3/4$ (same as part (b)).

(d) $\boxed{\text{Var}(X) = E(X^2) - E(X)^2 = 3/4 - 1/4 = 1/2.}$

29. Use $\text{Var}(X) = E(X^2) - E(X)^2 \Rightarrow 2 = E(X^2) - 25 \Rightarrow \boxed{E(X^2) = 27.}$

30. Make a table

X:	0	1
prob:	(1-p)	p
X^2	0	1.

From the table, $\boxed{E(X) = 0\cdot(1-p) + 1\cdot p = p.}$

Since X and X^2 have the same table $E(X^2) = E(X) = p$.

Therefore, $\boxed{\text{Var}(X) = p - p^2 = p(1-p).}$

31. Let X be the number of people who get their own hat.

Following the hint: let X_j represent whether person j gets their own hat. That is, $X_j = 1$ if person j gets their hat and 0 if not.

We have, $X = \sum_{j=1}^{100} X_j$, so $E(X) = \sum_{j=1}^{100} E(X_j)$.

Since person j is equally likely to get any hat, we have $P(X_j = 1) = 1/100$. Thus, $X_j \sim$ Bernoulli(1/100) $\Rightarrow E(X_j) = 1/100 \Rightarrow \boxed{E(X) = 1.}$

32. (a) It is easy to see that (e.g. look at the probability tree) $P(2^k) = \frac{1}{2^{k+1}}$.

(b) $E(X) = \sum_{k=0}^{\infty} 2^k \frac{1}{2^{k+1}} = \sum \frac{1}{2} = \infty$. Technically, $E(X)$ is undefined in this case.

(c) Technically, $E(X)$ is undefined in this case. But the value of ∞ tells us what is wrong with the scheme. Since the average last bet is infinite, I need to have an infinite amount of money in reserve.

This problem and solution is often referred to as the **St. Petersburg paradox**

5 Probability Mass Functions, Probability Density Functions and Cumulative Distribution Functions

33. For $y = 0, 2, 4, \ldots, 2n$,

$$P(Y = y) = P(X = \frac{y}{2}) = \binom{n}{y/2}\left(\frac{1}{2}\right)^n.$$

34. (a) We have $f_X(x) = 1$ for $0 \le x \le 1$. The cdf of X is

$$F_X(x) = \int_0^x f_X(t)dt = \int_0^x 1dt = x.$$

(b) Since X is between 0 and 1 we have Y is between 5 and 7. Now for $5 \le y \le 7$, we have

$$F_Y(y) = P(Y \le y) = P(2X + 5 \le y) = P(X \le \frac{y-5}{2}) = F_X(\frac{y-5}{2}) = \frac{y-5}{2}.$$

Differentiating $P(Y \le y)$ with respect to y, we get the probability density function of Y, for $5 \le y \le 7$,

$$f_Y(y) = \frac{1}{2}.$$

35. **(a)** We have cdf of X,

$$F_X(x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}.$$

Now for $y \geq 0$, we have

(b)

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = 1 - e^{-\lambda\sqrt{y}}.$$

Differentiating $F_Y(y)$ with respect to y, we have

$$f_Y(y) = \frac{\lambda}{2} y^{-\frac{1}{2}} e^{-\lambda\sqrt{y}}.$$

36. **(a)** We first make the probability tables

X	0	2	3
prob.	0.3	0.1	0.6
Y	3	3	12

$\Rightarrow\ E(X) = 0 \cdot 0.3 + 2 \cdot 0.1 + 3 \cdot 0.6 = 2$

(b) $E(X^2) = 0 \cdot 0.3 + 4 \cdot 0.1 + 9 \cdot 0.6 = 5.8 \ \Rightarrow\ \text{Var}(X) = E(X^2) - E(X)^2 = 5.8 - 4 = 1.8.$

(c) $E(Y) = 3 \cdot 0.3 + 3 \cdot 0.1 + 12 \cdot 6 = 8.4.$

(d) From the table we see that $F_Y(7) = P(Y \leq 7) = 0.4.$

37. **(a)** There are a number of ways to present this.
Let T be the total number of times you roll a 6 in the 100 rolls. We know $T \sim \text{Binomial}(100, 1/6)$. Since you win \$3 every time you roll a 6, we have $X = 3T$. So, we can write

$$P(X = 3k) = \binom{100}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{100-k}, \quad \text{for } k = 0,\ 1,\ 2,\ \ldots,\ 100.$$

Alternatively we could write

$$P(X = x) = \binom{100}{x/3}\left(\frac{1}{6}\right)^{x/3}\left(\frac{5}{6}\right)^{100-x/3}, \quad \text{for } x = 0,\ 3,\ 6,\ \ldots,\ 300.$$

(b) $E(X) = E(3T) = 3E(T) = 3 \cdot 100 \cdot \frac{1}{6} = 50,$
$\text{Var}(X) = \text{Var}(3T) = 9\text{Var}(T) = 9 \cdot 100 \cdot \frac{1}{6} \cdot \frac{5}{6} = 125.$

(c) (i) Let T_1 be the total number of times you roll a 6 in the first 25 rolls. So, $X_1 = 3T_1$ and $Y = 12T_1$.
Now, $T_1 \sim \text{Binomial}(25, 1/6)$, so

$$E(Y) = 12E(T_1) = 12 \cdot 25 \cdot 16 = 50.$$

and

$$\text{Var}(Y) = 144\text{Var}(T_1) = 144 \cdot 25 \cdot \frac{1}{6} \cdot \frac{5}{6} = 500.$$

(ii) The expectations are the same by linearity because X and Y are the both

$3 \times 100 \times$ a Bernoulli(1/6) random variable.

For the variance, $\text{Var}(X) = 4\text{Var}(X_1)$ because X is the sum of 4 *independent* variables all identical to X_1. However $\text{Var}(Y) = \text{Var}(4X_1) = 16\text{Var}(X_1)$. So, the variance of Y is 4 times that of X. This should make some intuitive sense because X is built out of more independent trials than X_1.

Another way of thinking about it is that the difference between Y and its expectation is four times the difference between X_1 and its expectation. However, the difference between X and its expectation is the sum of such a difference for X_1, X_2, X_3, and X_4. Its probably the case that some of these deviations are positive and some are negative, so the absolute value of this difference for the sum is probably less than four times the absolute value of this difference for one of the variables. (I.e., the deviations are likely to cancel to some extent.)

38. The CDF for R is

$$F_R(r) = P(R \le r) = \int_0^r 2e^{-2u}\, du = -e^{-2u}\Big|_0^r = 1 - e^{-2r}.$$

Next, we find the CDF of T. T takes values in $(0, \infty)$.
For $0 < t$,

$$F_T(t) = P(T \le t) = P(1/R < t) = P(1/t > R) = 1 - F_R(1/t) = e^{-2/t}.$$

We differentiate to get $f_T(t) = \dfrac{d}{dt}\left(e^{-2/t}\right) = \dfrac{2}{t^2}e^{-2/t}$

39. First we find the value of a:

$$\int_0^1 f(x)\, dx = 1 = \int_0^1 x + ax^2\, dx = \frac{1}{2} + \frac{a}{3} \;\Rightarrow\; a = 3/2.$$

The CDF is $F_X(x) = P(X \le x)$. We break this into cases:
i) $b < 0 \;\Rightarrow\; F_X(b) = 0$.
ii) $0 \le b \le 1 \;\Rightarrow\; F_X(b) = \displaystyle\int_0^b x + \frac{3}{2}x^2\, dx = \frac{b^2}{2} + \frac{b^3}{2}$.
iii) $1 < x \;\Rightarrow\; F_X(b) = 1$.
Using F_X we get

$$P(.5 < X < 1) = F_X(1) - F_X(.5) = 1 - \left(\frac{.5^2 + .5^3}{2}\right) = \frac{13}{16}.$$

40. (**PMF of a sum**) First we'll give the joint probability table:

Y\X	0	1	
0	1/3	1/3	2/3
1	1/6	1/6	1/3
	1/2	1/2	1

We'll use the joint probabilities to build the probability table for the sum.

$X+Y$	0	1	2
(X,Y)	(0,0)	(0,1), (1,0)	(1,1)
prob.	1/3	$1/6+1/3$	1/6
prob.	1/3	1/2	1/6

41. **(a)** Note: $Y=1$ when $X=1$ or $X=-1$, so

$$P(Y=1)=P(X=1)+P(X=-1).$$

Values y of Y	0	1	4
pmf $p_Y(y)$	3/15	6/15	6/15

(b) and **(c)** To distinguish the distribution functions we'll write F_x and F_Y.

Using the tables in part (a) and the definition $F_X(a)=P(X\le a)$ etc. we get

a	-1.5	3/4	7/8	1	1.5	5
$F_X(a)$	1/15	6/15	6/15	10/15	10/15	1
$F_Y(a)$	0	3/15	3/15	9/15	9/15	1

42. The jumps in the distribution function are at 0, 2, 4. The value of $p(a)$ at a jump is the height of the jump:

a	0	2	4
$p(a)$	1/5	1/5	3/5

43.
(i) yes, discrete, (ii) no, (iii) no, (iv) no, (v) yes, continuous
(vi) no (vii) yes, continuous, (viii) yes, continuous.

44. $P(1/2\le X\le 3/4)=F(3/4)-F(1/2)=(3/4)^2-(1/2)^2=\boxed{5/16}$.

45. **(a)** $P(1/4\le X\le 3/4)=F(3/4)-F(1/4)=\boxed{11/16=.6875.}$

(b) $f(x)=F'(x)=4x-4x^3$ in [0,1].

6 Distributions with Names

Exponential Distribution

46. We compute

$$P(X\ge 5)=1-P(X<5)=1-\int_0^5 \lambda \mathrm{e}^{-\lambda x}dx=1-(1-\mathrm{e}^{-5\lambda})=\mathrm{e}^{-5\lambda}.$$

(b) We want $P(X\ge 15|X\ge 10)$. First observe that $P(X\ge 15, X\ge 10)=P(X\ge 15)$. From similar computations in (a), we know

$$P(X\ge 15)=\mathrm{e}^{-15\lambda} \qquad P(X\ge 10)=\mathrm{e}^{-10\lambda}.$$

From the definition of conditional probability,

$$P(X \geq 15|X \geq 10) = \frac{P(X \geq 15, X \geq 10)}{P(X \geq 10)} = \frac{P(X \geq 15)}{P(X \geq 10)} = \mathrm{e}^{-5\lambda}$$

Note: This is an illustration of the memorylessness property of the exponential distribution.

47. Normal Distribution: (a) We have

$$F_X(x) = P(X \leq x) = P(3Z + 1 \leq x) = P(Z \leq \frac{x-1}{3}) = \Phi\left(\frac{x-1}{3}\right).$$

(b) Differentiating with respect to x, we have

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{dx}} F_X(x) = \frac{1}{3}\phi\left(\frac{x-1}{3}\right).$$

Since $\phi(x) = (2\pi)^{-\frac{1}{2}}\mathrm{e}^{-\frac{x^2}{2}}$, we conclude

$$f_X(x) = \frac{1}{3\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-1)^2}{2\cdot 3^2}},$$

which is the probability density function of the $N(1, 9)$ distribution. **Note:** The arguments in (a) and (b) give a proof that $3Z+1$ is a normal random variable with mean 1 and variance 9. See Problem Set 3, Question 5.

(c) We have

$$P(-1 \leq X \leq 1) = P\left(-\frac{2}{3} \leq Z \leq 0\right) = \Phi(0) - \Phi\left(-\frac{2}{3}\right) \approx 0.2475$$

(d) Since $E(X) = 1$, $\mathrm{Var}(X) = 9$, we want $P(-2 \leq X \leq 4)$. We have

$$P(-2 \leq X \leq 4) = P(-3 \leq 3Z \leq 3) = P(-1 \leq Z \leq 1) \approx 0.68.$$

48. Transforming Normal Distributions
(a) Note, Y follows what is called a *log-normal distribution.*
$F_Y(a) = P(Y \leq a) = P(e^Z \leq a) = P(Z \leq \ln(a)) = \boxed{\Phi(\ln(a)).}$
Differentiating using the chain rule:

$$f_y(a) = \frac{d}{da}F_Y(a) = \frac{d}{da}\Phi(\ln(a)) = \frac{1}{a}\phi(\ln(a)) = \boxed{\frac{1}{\sqrt{2\pi}\,a}\mathrm{e}^{-(\ln(a))^2/2}.}$$

(b) (i) The 0.33 quantile for Z is the value $q_{0.33}$ such that $P(Z \leq q_{0.33}) = 0.33$. That is, we want

$$\Phi(q_{0.33}) = 0.33 \;\Leftrightarrow\; \boxed{q_{0.33} = \Phi^{-1}(0.33)}.$$

(ii) We want to find $q_{0.9}$ where

$$F_Y(q_{0.9}) = 0.9 \;\Leftrightarrow\; \Phi(\ln(q_{0.9})) = 0.9 \;\Leftrightarrow\; \boxed{q_{0.9} = \mathrm{e}^{\Phi^{-1}(0.9)}}.$$

(iii) As in (ii) $q_{0.5} = \mathrm{e}^{\Phi^{-1}(0.5)} = \mathrm{e}^0 = \boxed{1}$.

49. (**Random variables derived from normal r.v.**)
(a) $\mathrm{Var}(X_j) = 1 = E(X_j^2) - E(X_j)^2 = E(X_j^2)$. QED

(b) $E(X_j^4) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} x^4 \mathrm{e}^{-x^2/2}\, dx.$

(Extra credit) By parts: let $u = x^3$, $v' = x\mathrm{e}^{-x^2/2} \;\Rightarrow\; u' = 3x^2$, $v = -\mathrm{e}^{-x^2/2}$

$E(X_j^4) = \dfrac{1}{\sqrt{2\pi}} \left[x^3 \mathrm{e}^{-x^2/2} \Big|_{infty}^{\infty} + \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} 3x^2 \mathrm{e}^{-x^2/2}\, dx \right]$

The first term is 0 and the second term is the formula for $3E(X_j^2) = 3$ (by part (a)). Thus, $E(X_j^4) = 3$.

(c) $\mathrm{Var}(X_j^2) = E(X_j^4) - E(X_j^2)^2 = 3 - 1 = 2$. QED

(d) $E(Y_{100}) = E(100X_j^2) = 100$. $\mathrm{Var}(Y_{100}) = 100\mathrm{Var}(X_j) = 200$.

The CLT says Y_{100} is approximately normal. Standardizing gives

$P(Y_{100} > 110) = P\left(\dfrac{Y_{100} - 100}{\sqrt{200}}) > \dfrac{10}{\sqrt{200}} \right) \approx P(Z > 1/\sqrt{2}) = \boxed{0.24}$.

This last value was computed using R: `1 - pnorm(1/sqrt(2),0,1)`.

50. More Transforming Normal Distributions
(a) Let $\phi(z)$ and $\Phi(z)$ be the PDF and CDF of Z.

$F_Y(y) = P(Y \le y) = P(aZ + b \le y) = P(Z \le (y - b)/a) = \Phi((y - b)/a)$.

Differentiating:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\Phi((y - b)/a) = \frac{1}{a}\phi((y - b)/a) = \frac{1}{\sqrt{2\pi}\, a}\, \mathrm{e}^{-(y-b)^2/2a^2}.$$

Since this is the density for $\mathrm{N}(a, b)$ we have shown $Y \sim \mathrm{N}(a, b)$.

(b) By part (a), $Y \sim \mathrm{N}(\mu, \sigma) \;\Rightarrow\; Y = \sigma Z + \mu$.

But, this implies $(Y - \mu)/\sigma = Z \sim \mathrm{N}(0, 1)$. QED

51. (**Sums of normal random variables**)
(a) $E(W) = 3E(X) - 2E(Y) + 1 = 6 - 10 + 1 = \boxed{-3}$

$\mathrm{Var}(W) = 9\mathrm{Var}(X) + 4\mathrm{Var}(Y) = 45 + 36 = \boxed{81}$

(b) Since the sum of independent normal is normal part (a) shows: $W \sim N(-3, 81)$. Let $Z \sim N(0, 1)$. We *standardize* W: $P(W \le 6) = P\left(\dfrac{W + 3}{9} \le \dfrac{9}{9}\right) = P(Z \le 1) \approx \boxed{.84.}$

52.

Method 1

$U(a, b)$ has density $f(x) = \dfrac{1}{b - a}$ on $[a, b]$. So,

$$E(X) = \int_a^b x f(x)\, dx = \frac{1}{b-a}\int_a^b x\, dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{b^2-a^2}{2(b-a)} = \boxed{\frac{a+b}{2}.}$$
$$E(X^2) = \int_a^b x^2 f(x)\, dx = \frac{1}{b-a}\int_a^b x^2\, dx = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)}.$$

Finding $\mathrm{Var}(X)$ now requires a little algebra,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{b^3-a^3}{3(b-a)} - \frac{(b+a)^2}{4}$$
$$= \frac{4(b^3-a^3) - 3(b-a)(b+a)^2}{12(b-a)} = \frac{b^3 - 3ab^2 + 3a^2b - a^3}{12(b-a)} = \frac{(b-a)^3}{12(b-a)} = \boxed{\frac{(b-a)^2}{12}.}$$

Method 2

There is an easier way to find $E(X)$ and $\mathrm{Var}(X)$.

Let $U \sim \mathrm{U}(a,b)$. Then the calculations above show $E(U) = 1/2$ and $(E(U^2) = 1/3$ $\Rightarrow$ $\mathrm{Var}(U) = 1/3 - 1/4 = 1/12$.

Now, we know $X = (b-a)U + a$, so $E(X) = (b-a)E(U) + a = (b-a)/2 + a = (b+a)/2$ and $\mathrm{Var}(X) = (b-a)^2\mathrm{Var}(U) = (b-a)^2/12$.

53. **(a)** $S_n \sim$ Binomial(n, p), since it is the number of successes in n independent Bernoulli trials.

(b) $T_m \sim$ Binomial(m, p), since it is the number of successes in m independent Bernoulli trials.

(c) $S_n + T_m \sim$ Binomial($n+m$, p), since it is the number of successes in $n+m$ independent Bernoulli trials.

(d) Yes, S_n and T_m are independent. We haven't given a formal definition of independent random variables yet. But, we know it means that knowing S_n gives no information about T_m. This is clear since the first n trials are independent of the last m.

54. The density for this distribution is $f(x) = \lambda\,\mathrm{e}^{-\lambda x}$. We know (or can compute) that the distribution function is $F(a) = 1 - \mathrm{e}^{-\lambda a}$. The median is the value of a such that $F(a) = .5$. Thus, $1 - \mathrm{e}^{-\lambda a} = 0.5 \Rightarrow 0.5 = \mathrm{e}^{-\lambda a} \Rightarrow \log(0.5) = -\lambda a \Rightarrow \boxed{a = \log(2)/\lambda.}$

55. **(a)** $P(X > a) = \displaystyle\int_a^\infty \frac{\alpha\, m^\alpha}{x^{\alpha+1}} = \left. -\frac{m^\alpha}{x^\alpha}\right|_a^\alpha = \boxed{\frac{m^\alpha}{a^\alpha}.}$

(b) We want the value $q_{.8}$ where $P(X \le q_{.8}) = 0.8$.

This is equivalent to $P(X > q_{.8}) = 0.2$. Using part (a) and the given values of m and α we have $\dfrac{1}{q_{.8}} = .2 \Rightarrow \boxed{q_{.8} = 5.}$

7 Joint Probability, Covariance, Correlation

56. (**Another Arithmetic Puzzle**)

(a) $S = X + Y$ takes values 0, 1, 2 and $T = X - Y$ takes values -1, 0, 1.

First we make two tables: the joint probability table for X and Y and a table given the values (S,T) corresponding to values of (X,Y), e.g. $(X,Y) = (1,1)$ corresponds to $(S,T) = (2,0)$.

$X\backslash Y$	0	1
0	1/4	1/4
1	1/4	1/4

Joint probabilities of X and Y

$X\backslash Y$	0	1
0	0,0	0,-1
1	1,1	2,0

Values of (S,T) corresponding to X and Y

We can use the two tables above to write the joint probability table for S and T. The marginal probabilities are given in the table.

$S\backslash T$	-1	0	1	
0	1/4	1/4	0	1/2
1	0	0	1/4	1/4
2	0	0	1/4	1/4
	1/4	1/4	1/2	1

Joint and marginal probabilities of S and T

(b) No probabilities in the table are the product of the corresponding marginal probabilities. (This is easiest to see for the 0 entries.) So, S and T are not independent

57. **(a)** The joint distribution is found by dividing each entry in the data table by the total number of people in the sample. Adding up all the entries we get 1725. So the joint probability table with marginals is

$Y\backslash X$	1	2	3	
1	$\frac{234}{1725}$	$\frac{225}{1725}$	$\frac{84}{1725}$	$\frac{543}{1725}$
2	$\frac{180}{1725}$	$\frac{453}{1725}$	$\frac{161}{1725}$	$\frac{794}{1725}$
3	$\frac{39}{1725}$	$\frac{192}{1725}$	$\frac{157}{1725}$	$\frac{388}{1725}$
	$\frac{453}{1725}$	$\frac{839}{1725}$	$\frac{433}{1725}$	1

The marginal distribution of X is at the right and of Y is at the bottom.

(b) X and Y are dependent because, for example,

$$P(X = 1 \text{ and } Y = 1) = \frac{234}{1725} \approx 0.136$$

is not equal to

$$P(X = 1)P(Y = 1) = \frac{453}{1725} \cdot \frac{543}{1725} \approx 0.083.$$

58. **(a)** Total probability must be 1, so

$$1 = \int_0^3 \int_0^3 f(x,y)\, dy\, dx = \int_0^3 \int_0^3 c(x^2y + x^2y^2)\, dy\, dx = c \cdot \frac{243}{2},$$

(Here we skipped showing the arithmetic of the integration) Therefore, $c = \frac{2}{243}$.

(b)

$$\begin{aligned} P(1 \le X \le 2,\, 0 \le Y \le 1) &= \int_1^2 \int_0^1 f(x,y)\,dy\,dx \\ &= \int_1^2 \int_0^1 c(x^2y + x^2y^2)\,dy\,dx \\ &= c \cdot \frac{35}{18} \\ &= \frac{70}{4374} \approx 0.016 \end{aligned}$$

(c) For $0 \le a \le 3$ and $0 \le b \le 3$. we have

$$F(a,b) = \int_0^a \int_0^b f(x,y) dy\,dx = c\left(\frac{a^3b^2}{6} + \frac{a^3b^3}{9}\right)$$

(d) Since $y = 3$ is the maximum value for Y, we have

$$F_X(a) = F(a,3) = c\left(\frac{9a^3}{6} + 3a^3\right) = \frac{9}{2}c\,a^3 = \frac{a^3}{27}$$

(e) For $0 \le x \le 3$, we have, by integrating over the entire range for y,

$$f_X(x) = \int_0^3 f(x,y)\,dy = cx^2\left(\frac{3^2}{2} + \frac{3^3}{3}\right) = c\frac{27}{2}x^2 = \frac{1}{9}x^2.$$

This is consistent with (c) because $\frac{d}{dx}(x^3/27) = x^2/9$.

(f) Since $f(x,y)$ separates into a product as a function of x times a function of y we know X and Y are independent.

59. **(a)** First note by linearity of expectation we have $E(X+s) = E(X) + s$, thus $X + s - E(X+s) = X - E(X)$.

Likewise $Y + u - E(Y+u) = Y - E(Y)$.

Now using the definition of covariance we get

$$\begin{aligned} \text{Cov}(X+s, Y+u) &= E((X + s - E(X+s)) \cdot (Y + u - E(Y+u))) \\ &= E((X - E(X)) \cdot (Y - E(Y))) \\ &= \text{Cov}(X,Y). \end{aligned}$$

(b) This is very similar to part (a).

We know $E(rX) = rE(X)$, so $rX - E(rX) = r(X - E(X))$. Likewise $tY - E(tY) = s(Y - E(Y))$. Once again using the definition of covariance we get

$$\begin{aligned} \text{Cov}(rX, tY) &= E((rX - E(rX))(tY - E(tY))) \\ &= E(rt(X - E(X))(Y - E(Y))) \\ &\qquad \text{(Now we use linearity of expectation to pull out the factor of } rt\text{)} \\ &= rtE((X - E(X)(Y - E(Y)))) \\ &= rt\text{Cov}(X,Y) \end{aligned}$$

(c) This is more of the same. We give the argument with far fewer algebraic details

$$\begin{aligned}\text{Cov}(rX+s, tY+u) &= \text{Cov}(rX, tY) \text{ (by part (a))} \\ &= rt\text{Cov}(X,Y) \text{ (by part (b))}\end{aligned}$$

60. Using linearity of expectation, we have

$$\begin{aligned}\text{Cov}(X,Y) &= E\left[(X-E(X))(Y-E(Y))\right] \\ &= E\left[XY - E(X)Y - E(Y)X + E(X)E(Y)\right] \\ &= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y) \\ &= E(XY) - E(X)E(Y).\end{aligned}$$

61. (**Arithmetic Puzzle**) **(a)** The marginal probabilities have to add up to 1, so the two missing marginal probabilities can be computed: $P(X=3)=1/3$, $P(Y=3)=1/2$. Now each row and column has to add up to its respective margin. For example, $1/6+0+P(X=1, Y=3)=1/3$, so $P(X=1,Y=3)=1/6$. Here is the completed table.

$X \backslash Y$	1	2	3	
1	1/6	0	1/6	1/3
2	0	1/4	1/12	1/3
3	0	1/12	1/4	1/3
	1/6	1/3	1/2	1

(b) No, X and Y are not independent.
For example, $P(X=2,Y=1)=0 \neq P(X=2)\cdot P(Y=1)$.

62. (**Simple Joint Probability**) First we'll make the table for the joint pmf. Then we'll be able to answer the questions by summing up entries in the table.

$X \backslash Y$	1	2	3	4
1	2/80	3/80	4/80	5/80
2	3/80	4/80	5/80	6/80
3	4/80	5/80	6/80	7/80
4	5/80	6/80	7/80	8/80

(a) $P(X=Y) = p(1,1)+p(2,2)+p(3,3)+p(4,4) = \boxed{20/80 = 1/4.}$

(b) $P(XY=6) = p(2,3)+p(3,2) = \boxed{10/80 = 1/8.}$

(c) $P(1 \le X \le 2, 2 < Y \le 4)$ = sum of 4 red probabilities in the upper right corner of the table $= \boxed{20/80 = 1/4.}$

63. **(a)** X and Y are independent, so the table is computed from the product of the known marginal probabilities. Since they are independent, $\text{Cov}(X,Y)=0$.

$Y \backslash X$	0	1	P_Y
0	1/8	1/8	1/4
1	1/4	1/4	1/2
2	1/8	1/8	1/4
P_X	1/2	1/2	1

(b) The sample space is $\Omega = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$.
$P(X=0, F=0) = P(\{TTH, TTT\}) = 1/4$.
$P(X=0, F=1) = P(\{THH, THT\}) = 1/4$.
$P(X=0, F=2) = 0$.
$P(X=1, F=0) = 0$.
$P(X=1, F=1) = P(\{HTH, HTT\}) = 1/4$.
$P(X=1, F=2) = P(\{HHH, HHT\}) = 1/4$.

$F \backslash X$	0	1	P_F
0	1/4	0	1/4
1	1/4	1/4	1/2
2	0	1/4	1/4
P_X	1/2	1/2	1

$\text{Cov}(X, F) = E(XF) - E(X)E(F)$.
$E(X) = 1/2, \quad E(F) = 1, \; E(XF) = \sum x_i y_j p(x_i, y_j) = 3/4$.
$\Rightarrow \; \text{Cov}(X, F) = 3/4 - 1/2 = \boxed{1/4.}$

64. Covariance and Independence
(a)

$Y \backslash X$	-2	-1	0	1	2	
0	0	0	1/5	0	0	1/5
1	0	1/5	0	1/5	0	2/5
4	1/5	0	0	0	1/5	2/5
	1/5	1/5	1/5	1/5	1/5	1

Each column has only one nonzero value. For example, when $X = -2$ then $Y = 4$, so in the $X = -2$ column, only $P(X = -2, Y = 4)$ is not 0.

(b) Using the marginal distributions: $E(X) = \frac{1}{5}(-2 - 1 + 0 + 1 + 2) = 0$.
$E(Y) = 0 \cdot \dfrac{1}{5} + 1 \cdot \dfrac{2}{5} + 4 \cdot \dfrac{2}{5} = 2$.

(c) We show the probabilities don't multiply:
$P(X = -2, \, Y = 0) = 0 \neq P(X = -2) \cdot P(Y = 0) = 1/25$.
Since these are not equal X and Y are not independent. (It is obvious that X^2 is not independent of X.)

(d) Using the table from part (a) and the means computed in part (d) we get:

$$\begin{aligned}\text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ &= \frac{1}{5}(-2)(4) + \frac{1}{5}(-1)(1) + \frac{1}{5}(0)(0) + \frac{1}{5}(1)(1) + \frac{1}{5}(2)(4) \\ &= 0.\end{aligned}$$

65. Continuous Joint Distributions
(a) $F(a, b) = P(X \le a, Y \le b) = \displaystyle\int_0^a \int_0^b (x + y)\, dy\, dx$.

Inner integral: $\quad xy + \left.\dfrac{y^2}{2}\right|_0^b = xb + \dfrac{b^2}{2}$. Outer integral: $\quad \left.\dfrac{x^2}{2}b + \dfrac{b^2}{2}x\right|_0^a = \dfrac{a^2 b + ab^2}{2}$.

So $\boxed{F(x, y) = \dfrac{x^2 y + x y^2}{2}}$ and $\boxed{F(1, 1) = 1.}$

(b) $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy = xy + \left.\frac{y^2}{2}\right|_0^1 = \boxed{x + \frac{1}{2}.}$

By symmetry, $\boxed{f_Y(y) = y + 1/2.}$

(c) To see if they are independent we check if the joint density is the product of the marginal densities.

$f(x,y) = x + y, \quad f_X(x) \cdot f_Y(y) = (x + 1/2)(y + 1/2).$

Since these are not equal, $\boxed{X \text{ and } Y \text{ are not independent.}}$

(d) $E(X) = \int_0^1 \int_0^1 x(x+y)\,dy\,dx = \int_0^1 \left[x^2 y + x\left.\frac{y^2}{2}\right|_0^1 \right] dx = \int_0^1 x^2 + \frac{x}{2}\,dx = \boxed{\frac{7}{12}.}$

(Or, using (b), $E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(x+1/2)\,dx = 7/12.$)

By symmetry $\boxed{E(Y) = 7/12.}$

$E(X^2+Y^2) = \int_0^1 \int_0^1 (x^2+y^2)(x+y)\,dy\,dx = \boxed{\frac{5}{6}.}$

$E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dy\,dx = \boxed{\frac{1}{3}.}$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \frac{49}{144} = \boxed{-\frac{1}{144}.}$

8 Law of Large Numbers, Central Limit Theorem

66. Standardize:

$$\begin{aligned}
P\left(\sum_i X_i < 30\right) &= P\left(\frac{\frac{1}{n}\sum X_i - \mu}{\sigma/\sqrt{n}} < \frac{30/n - \mu}{\sigma/\sqrt{n}}\right) \\
&\approx P\left(Z < \frac{30/100 - 1/5}{1/30}\right) \quad \text{(by the central limit theorem)} \\
&= P(Z < 3) \\
&= 0.9987 \ \text{(from the table of normal probabilities)}
\end{aligned}$$

67. All or None

If $p < .5$ your expected winnings on any bet is negative, if $p = .5$ it is 0, and if $p > .5$ is is positive. By making a lot of bets the minimum strategy will 'win' you close to the expected average. So if $p \le .5$ you should use the maximum strategy and if $p > .5$ you should use the minumum strategy.

68. (**Central Limit Theorem**) Let $T = X_1 + X_2 + \ldots + X_{81}$. The central limit theorem says that

$$T \approx \text{N}(81 * 5, 81 * 4) = \text{N}(405, 18^2)$$

Standardizing we have

$$\begin{aligned} P(T > 369) &= P\left(\frac{T-405}{18} > \frac{369-405}{18}\right) \\ &\approx P(Z > -2) \\ &\approx 0.975 \end{aligned}$$

The value of 0.975 comes from the rule-of-thumb that $P(|Z| < 2) \approx 0.95$. A more exact value (using R) is $P(Z > -2) \approx 0.9772$.

69. (**Binomial $\approx$ normal**)
$X \sim \text{binomial}(100, 1/3)$ means X is the sum of 100 i.i.d. Bernoulli(1/3) random variables X_i.

We know $E(X_i) = 1/3$ and $\text{Var}(X_i) = (1/3)(2/3) = 2/9$. Therefore the central limit theorem says

$$X \approx \text{N}(100/3, 200/9)$$

Standardization then gives

$$P(X \le 30) = P\left(\frac{X - 100/3}{\sqrt{200/9}} \le \frac{30 - 100/3}{\sqrt{200/9}}\right) \approx P(Z \le -0.7071068) \approx 0.239751$$

We used R to do these calculations The approximation agrees with the 'exact' number to 2 decimal places.

70. (**More Central Limit Theorem**)
Let X_j be the IQ of a randomly selected person. We are given $E(X_j) = 100$ and $\sigma_{X_j} = 15$.
Let $\overline{X}$ be the average of the IQ's of 100 randomly selected people. Then we know

$$E(\overline{X}) = 100 \quad \text{and} \quad \sigma_{\overline{X}} = 15/\sqrt{100} = 1.5.$$

The problem asks for $P(\overline{X} > 115)$. Standardizing we get $P(\overline{X} > 115) \approx P(Z > 10)$. This is effectively 0.

9 R Problems

R will not be on the exam. However, these problems will help you understand the concepts we've been studying.

71. R simulation
(a) $E(X_j) = 0 \Rightarrow E(\overline{X}_n) = 0$.
$\text{Var}(X_j) = 1 \Rightarrow \text{Var}\left(\frac{X_1 + \ldots + X_n}{n}\right) = \frac{1}{n} \Rightarrow \sigma_{\overline{X}_n} = \frac{1}{\sqrt{n}}$.

(b) Here's my R code:

```
x = rnorm(100*1000,0,1)
data = matrix(x, nrow=100, ncol=1000)
data1 = data[1,]
```

```
m1 = mean(data1)
v1 = var(data1)
data9 = colMeans(data[1:9,])
m9 = mean(data9)
v9 = var(data9)
data100 = colMeans(data)
m100 = mean(data100)
v100 = var(data100)
#display the results
print(m1)
print(v1)
print(m9)
print(v9)
print(m100)
print(v100)
```

Note if $x = [x_1, x_2, \ldots, x_n]$ then `var(x)` actually computes $\frac{\sum x_k}{n-1}$ instead of $\frac{\sum x_k}{n}$. There is a good reason for this which we will learn in the statistics part of the class. For now, it's enough to note that if $n = 1000$ the using n or $n-1$ won't make much difference.

72. R Exercise

```
a = runif(5*1000,0,1)
data = matrix(a,5,1000)
x = colSums(data[1:3,])
y = colSums(data[3:5,])
print(cov(x,y))
```

Extra Credit

Method 1 (Algebra)

First, if $i \neq j$ we know X_i and X_j are independent, so $\text{Cov}(X_i, X_j) = 0$.

$$\begin{aligned}
\text{Cov}(X, Y) &= \text{Cov}(X_1 + X_2 + X_3, X_3 + X_4 + X_5) \\
&= \text{Cov}(X_1, X_3) + \text{Cov}(X_1, X_4) + \text{Cov}(X_1, X_5) \\
&\qquad + \text{Cov}(X_2, X_3) + \text{Cov}(X_2, X_4) + \text{Cov}(X_2, X_5) \\
&\qquad + \text{Cov}(X_3, X_3) + \text{Cov}(X_3, X_4) + \text{Cov}(X_3, X_5) \\
&\quad \text{(most of these terms are 0)} \\
&= \text{Cov}(X_3, X_3) \\
&= \text{Var}(X_3) \\
&= \frac{1}{12} \qquad \text{(known variance of a uniform(0,1) distribution)}
\end{aligned}$$

Method 2 (Multivariable calculus)

In 5 dimensional space we have the joint distribution

$$f(x_1, x_2, x_3, x_4, x_5) = 1.$$

Computing directly

$$E(X) = E(X_1 + X_2 + X_3) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (x_1 + x_2 + x_3)\, dx_1\, dx_2\, dx_3, dx_4\, dx_5$$

$$\text{first integral} = \frac{1}{2} + x_2 + x_3$$

$$\text{second integral} = \frac{1}{2} + \frac{1}{2} + x_3 = 1 + x_3$$

$$\text{third integral} = \frac{3}{2}$$

$$\text{fourth integral} = \frac{3}{2}$$

$$\text{fifth integral} = \frac{3}{2}$$

So, $E(X) = 3/2$, likewise $E(Y) = 3/2$.

$$\begin{aligned} E(XY) &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (x_1 + x_2 + x_3)(x_3 + x_4 + x_5)\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5 \\ &= 7/3. \end{aligned}$$

$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{12} = .08333.$