18.04 Recitation 4
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1. We will compute $I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$ using Cauchy's integral formula. It will be helpful to recall the triangle inequality for integrals: $\left|\int_\Gamma f(z)dz\right| \le \int_\Gamma |f(z)||dz|$.

1.1. Consider the semicircle C in the upper half plane which is centered at 0 and has radius R. Use Cauchy's integral formula to compute $\int_C \frac{1}{(1+z^2)^2} dz$.

1.2. Decompose $C = C_1 \cup C_2$, where C_1 denotes the segment between $-R$ and R on the x-axis, and C_2 denotes the remaining part of C. Use the triangle inequality for integrals to give an upper bound on $\left|\int_{C_2} \frac{1}{(1+z^2)^2} dz\right|$.

1.3. Use the results of the previous two parts to obtain an estimate $\int_{C_1} \frac{1}{(1+z^2)^2} dz$. What happens as you take $R \to \infty$?

2.1. *(Cauchy's inequality)* Let C_R be the circle of radius R centered at the point z_0, and suppose that f is analytic on C_R and its interior. Further, let $M_R = \max_{z \in C_R} |f(z)|$. Use Cauchy's integral formula for derivatives, and the triangle inequality for integrals to show that

$$\left|f^{(n)}(z_0)\right| \le \frac{n! M_R}{R^n}.$$

2.2. *(Liouville's Theorem)* Now, suppose f is an entire function and $|f(z)| \le M$ for all $z \in \mathbb{C}$. By analyzing the $n = 1$ case in the previous part, what can you say about f?

3. *(Fundamental Theorem of Algebra)* Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a degree n polynomial with $a_n \ne 0$. We will show that $P(z)$ has exactly n roots (counting multiplicities) over $\mathbb{C}$.

3.1. Assume for contradiction that $P(z) \ne 0$ for all $z \in \mathbb{C}$. Show that under this assumption, $f(z) := 1/P(z)$ is entire and bounded. Use Liouville's theorem to get a contradiction.

3.2. The previous part shows that P must have at least one root. Iterate it to show that P has exactly n roots (counting multiplicites).

4. *(Mean value property)* Let C_R be the circle of radius R centered at the point z_0, and suppose that f is analytic on C_R and its interior. Use Cauchy's integral formula to show that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta.$$