18.04 Recitation 11
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1.1. Find an LFT from the half-plane $H_\alpha := \{(x, y) : y > x\tan(\alpha)\}$ to the unit disc D_1 centered at the origin.

Ans: First, rotate by α clockwise to map H_α to the upper half-plane H. Then, use $T(z) = \frac{z-i}{z+i}$ to the map H to the unit-disc.

1.2. Find a conformal map from the strip $I_\pi := \{(x, y) : 0 < y < \pi\}$ to the upper half-plane H.

Ans: e^z.

1.3. Find a conformal map from the upper semi-disc $R_2 := \{(x, y) \in D_1 : y > 0\}$ to the upper half-plane H.

Ans: First, map $z_1 = 1$, $z_2 = i$ and $z_3 = -1$ to the three points $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$. This can be accomplished by the LFT

$$\begin{aligned} T_2(z) &= \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1} \\ &= \frac{(z-1)(i+1)}{(z+1)(i-1)} \\ &= -i\frac{z-1}{z+1}. \end{aligned}$$

Since $T_2(0) = i$, it follows that the segment $(-1, 1)$ on the real axis is mapped to the positive imaginary axis. It follows that T_2 maps the upper semi-disc to the first quadrant $Q_1 = \{(x, y) : x > 0, y > 0\}$. Now, use z^2 to map Q_1 to H.

1.4. Find a conformal map from the "infinite well" $W_\pi := \{(x, y) : 0 < y < \pi, x < 0\}$ to the upper half-plane.

Ans: First, use e^z to map W_π to the upper semi-disk R_2. Next, use the map from 2.3. to map R_2 to the upper half-plane.

2.1 Find the reflection of a point z_1 in the x-axis.

Ans: $\overline{z_1}$.

2.2. Define the reflection $r_C(z_2)$ of a point z_2 in a circle C as follows. Let T_{CL} be an LFT mapping the circle C to a line L. Then, $r_C(z_2) := T_{CL}^{-1}(r_L(T_{CL}(z_2)))$, where r_L denotes reflection in the line L. Use this definition to find the reflection of a point z_2 in the unit circle.

Ans: We know that the LFT

$$T^{-1}(z) = \frac{z-i}{z+i}$$

maps the x-axis to the unit circle. Therefore,

$$T(z) = i\frac{z+1}{-z+1}$$

maps the unit circle to the x-axis. Computing the above expression directly now gives that the reflection of z_2 in the unit circle is $\frac{1}{\overline{z_2}}$, which is what we expect.