18.04 Practice problems exam 1, Spring 2018

Problem 1. Complex arithmetic
(a) Find the real and imaginary part of $\dfrac{z+2}{z-1}$.

(b) Solve $z^4 - i = 0$.

(c) Find all possible values of $\sqrt{\sqrt{i}}$.

(d) Express $\cos(4x)$ in terms of $\cos(x)$ and $\sin(x)$.

(e) When does equality hold in the triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$?

(f) Draw a picture illustrating the polar coordinates of z and $1/z$.

Problem 2. Functions
(a) Show that $\sinh(z) = -i\sin(iz)$.

(b) Give the real and imaginary part of $\cos(z)$ in terms of x and y using regular and hyperbolic sin and cos.

(c) Is it true that $|a^b| = |a|^{|b|}$?

Problem 3. Mappings
(a) Show that the function $f(z) = \frac{z-i}{z+i}$ maps the upper half plane to the unit disk.
(i) Show it maps the real axis to the unit circle.
(ii) Show it maps i to 0.
(iii) Conclude that the upper half plane is mapped to the unit disk.

(b) Show that the function $f(z) = \dfrac{z+2}{z-1}$ maps the unit circle to the line $x = -1/2$.

Problem 4. Analytic functions
(a) Show that $f(z) = \mathrm{e}^z$ is analytic using the Cauchy Riemann equations.

(b) Show that $f(z) = \overline{z}$ is not analytic.

(c) Give a region in the z-plane for which $w = z^3$ is a one-to-one map onto the entire w-plane.

(d) Choose a branch of $z^{1/3}$ and a region of the z-plane where this branch is analytic. Do this so that the image under $z^{1/3}$ is contained in your region from part (c).

Problem 5. Line integrals
(a) Compute $\displaystyle\int_C x\,dz$, where C is the unit square.

(b) Compute $\displaystyle\int_C \frac{1}{|z|}\,dz$, where C is the unit circle.

(c) Compute $\displaystyle\int_C z\cos(z^2)\,dz$, where C is the unit circle.

(d) Draw the region $\mathbf{C} - \{x + i\sin(x) \text{ for } x \geq 0\}$. Is this region simply connected? Could you define a branch of log on this region?

(e) Compute $\int_C \frac{z^2}{z^4-1}$ over the circle of radius 3 with center 0.

(f) Does $\displaystyle\int_C \frac{\mathrm{e}^z}{z^2}\,dz = 0$?. Here C is a simple closed curve.

(g) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+16}\,dx$.

Problem 6.
Suppose $f(z)$ is entire and $|f(z)| > 1$ for all z. Show that f is a constant.

Problem 7.
Suppose $f(z)$ is analytic and $|f|$ is constant on the disk $|z - z_0| \le r$. Show that f is constant on the disk.

Extra problems from pset 4

Problem 8. (a) Let $f(z) = e^{\cos(z)} z^2$. Let A be the disk $|z - 5| \le 2$. Show that $f(z)$ attains both its maximum and minimum modulus in A on the circle $|z - 5| = 2$.

Hint: Consider $1/f(z)$.

(b) Suppose $f(z)$ is entire. Show that if $f^{(4)}(z)$ is bounded in the whole plane then $f(z)$ is a polynomial of degree at most 4.

(c) The function $f(z) = 1/z^2$ goes to 0 as $z \to \infty$, but it is not constant. Does this contradict Liouville's theorem?

Problem 9.
Show $\displaystyle\int_0^{\pi} e^{\cos\theta} \cos(\sin(\theta))\, d\theta = \pi$. Hint, consider e^z/z over the unit circle.

Problem 10.
(a) Suppose $f(z)$ is analytic on a simply connected region A and γ is a simple closed curve in A.. Fix z_0 in A, but not on γ. Use the Cauchy integral formulas to show that

$$\int_\gamma \frac{f'(z)}{z - z_0}\,dz = \int_\gamma \frac{f(z)}{(z - z_0)^2}\,dz.$$

(b) Challenge: Redo part (a), but drop the assumption that A is simply connected.

Problem 11.
(a) Compute $\displaystyle\int_C \frac{\cos(z)}{z}\,dz$, where C is the unit circle.

(b) Compute $\displaystyle\int_C \frac{\sin(z)}{z}\,dz$, where C is the unit circle.

(c) Compute $\displaystyle\int_C \frac{z^2}{z-1}\,dz$, where C is the circle $|z| = 2$.

(d) Compute $\displaystyle\int_C \frac{e^z}{z^2}\,dz$, where C is the circle $|z| = 1$.

(e) Compute $\displaystyle\int_C \frac{z^2-1}{z^2+1}\,dz$, where C is the circle $|z| = 2$.

(f) Compute $\displaystyle\int_C \frac{1}{z^2+z+1}\,dz$ where C is the circle $|z| = 2$.

Problem 12.
Suppose $f(z)$ is entire and $\displaystyle\lim_{z\to\infty} \frac{f(z)}{z} = 0$. Show that $f(z)$ is constant.

You may use Morera's theorem: if $g(z)$ is analytic on $A - \{z_0\}$ and continuous on A, then f is analytic on A.

18.04 Complex Variables with Applications
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