Answers to Problem Set Number 7 for 18.04. MIT (Fall 1999)

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1 Problems from the book by Saff and Snider.

1.1 Problem 04 in section 5.4.

Part (a): The series: $\displaystyle\sum_{j=1}^{\infty} \frac{z^j}{j^2}$. ________________

Clearly, the radius of convergence for this series is 1. Notice now that, for $|z| \leq 1$ we have $|z^j/j^2| \leq (1/j^2)$. Furthermore, the series of real numbers $\displaystyle\sum_{j=1}^{\infty} j^{-2}$ converges. Thus, by the Weierstrass M-test,

$$\sum_{j=1}^{\infty} \frac{z^j}{j^2} \text{ converges uniformly on the closed disc } |z| \leq 1\,.$$

That is: *in this example the series converges everywhere on its circle of convergence* $|z| = 1$.

Part (b): The series: $\displaystyle\sum_{j=1}^{\infty} \frac{z^j}{j}$. ________________

Again, the radius of convergence is 1. Now: when $z = 1$, the series $\displaystyle\sum_{j=1}^{\infty} j^{-1}$ does not converge (by the integral test), and when $z = -1$, the series $\displaystyle\sum_{j=1}^{\infty} (-1)^j\, j^{-1}$ does converge (by the alternating series test). Thus, *in this example the series converges for some points on its circle of convergence (at least for $z = -1$) and diverges for others (at least for $z = 1$).*

Part (d): The series: $\displaystyle\sum_{j=1}^{\infty} z^j$. ________________

Again, this series has a radius of convergence equal to 1. When $|z| = 1$, we notice that the *sequence* $|z^j|$ does not converge to 0 (since all of its terms are equal to 1). Therefore the series $\displaystyle\sum_{j=1}^{\infty} z^j$ cannot converge when $|z| = 1$. Thus, *in this example the series does not converge at any of the points on its circle of convergence.*

1.2 Problem 10 in section 5.4.

For the *Fibonacci numbers*, we know that $a_0 = a_1 = 1$ and

$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2\,. \tag{1.2.1}$$

Define now[1]

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

[1] The function f is known as the *generating* function for the sequence.

where the coefficients a_n are given by the Fibonacci sequence. Multiplying (1.2.1) by z^n and adding over n we obtain: $f(z) - a_0 - a_1 z = z f(z) - a_0 z + z^2 f(z)$. That is

$$1 + (z^2 + z - 1) f(z) = 0.$$

Let us double check this:

$$\begin{aligned}
1 + z f(z) + z^2 f(z) &= 1 + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} a_n z^{n+2} \\
&= 1 + \sum_{k=1}^{\infty} a_{k-1} z^k + \sum_{k=2}^{\infty} a_{k-2} z^k \\
&= 1 + a_0 z + \sum_{k=2}^{\infty} (a_{k-1} + a_{k-2}) z^k \\
&= 1 + z + \sum_{k=2}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k z^k = f(z)\,.
\end{aligned}$$

We can solve this for $f(z)$ and then do a partial fraction decomposition, to obtain

$$\begin{aligned}
f(z) &= \frac{1}{1 - z - z^2} = \frac{-1}{\left(z + \frac{1-\sqrt{5}}{2}\right)\left(z + \frac{1+\sqrt{5}}{2}\right)} = \frac{1/\sqrt{5}}{\left(z + \frac{1+\sqrt{5}}{2}\right)} + \frac{-1/\sqrt{5}}{\left(z + \frac{1-\sqrt{5}}{2}\right)} \\
&= \frac{2}{\sqrt{5}\left(1+\sqrt{5}\right)} \cdot \frac{1}{\left(1 + \frac{2}{(1+\sqrt{5})} z\right)} - \frac{2}{\sqrt{5}\left(1-\sqrt{5}\right)} \cdot \frac{1}{\left(1 + \frac{2}{(1-\sqrt{5})} z\right)} \\
&= -\frac{1}{\sqrt{5}} \cdot \frac{(1-\sqrt{5})}{2} \cdot \frac{1}{\left(1 - \left(\frac{1-\sqrt{5}}{2}\right) z\right)} + \frac{1}{\sqrt{5}} \cdot \frac{(1+\sqrt{5})}{2} \cdot \frac{1}{\left(1 - \left(\frac{1+\sqrt{5}}{2}\right) z\right)} \\
&= -\frac{1}{\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2} \sum_{n=0}^{\infty} \left(\frac{1-\sqrt{5}}{2}\right)^n z^n + \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2} \sum_{n=0}^{\infty} \left(\frac{1+\sqrt{5}}{2}\right)^n z^n \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] z^n\,,
\end{aligned}$$

which proves the desired result. Here we have used: the geometric series expansion and the fact that $\dfrac{2}{(1+\sqrt{5})} = -\dfrac{1-\sqrt{5}}{2}$.

1.3 Problem 02 in section 5.5.

The principal branch of $\sqrt{z}$ has a branch cut on the negative real axis, thus it is not analytic on any annulus around the origin. Therefore, no Laurent series expansion exists in $\mathbf{C} - \{0\}$. In fact

no branch of $\sqrt{z}$ has a Laurent series expansion in the domain $\mathbf{C} - \{0\}$.

1.4 Problem 05 in section 5.5.

To find the Laurent series of $\dfrac{(z+1)}{z(z-4)^3}$ in a punctured disc of radius 4 around $z_0 = 4$, let us rewrite this expression so that we can use the geometric series:

$$\begin{aligned}
\frac{(z+1)}{z(z-4)^3} &= \frac{1}{(z-4)^3}\left(1+\frac{1}{z}\right) \\
&= \frac{1}{(z-4)^3}\left(1+\frac{1}{4}\cdot\left(\frac{1}{1+\frac{z-4}{4}}\right)\right) \\
&= \frac{1}{(z-4)^3}\left(1+\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z-4}{-4}\right)^n\right) \\
&= \frac{1}{(z-4)^3} - \sum_{k=-3}^{\infty}\left(\frac{-1}{4}\right)^{k+4}(z-4)^k \,.
\end{aligned}$$

In the last line we have made the substitution $k = n - 3$.

1.5 Problem 06 in section 5.5.

To find the Laurent series for $z^2\cos\left(\dfrac{1}{3z}\right)$ in $|z| > 0$, we make use of the Taylor series for the cosine, which we know well:

$$\cos(\xi) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\xi^{2n} \,.$$

Thus

$$\begin{aligned}
z^2\cos\left(\frac{1}{3z}\right) &= z^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{(3z)^{2n}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{3^{2n}}z^{2(1-n)} \,.
\end{aligned}$$

2 Other problems.

2.1 Problem 7.1 in 1999.

Statement: Find a formula for

$$f(x) = \sum_{n=1}^{\infty}(-1)^n\frac{\sin(nx)}{n} \,, \tag{2.1.1}$$

where x is real and $-\pi \leq x \leq \pi$.

Write a short program to calculate and plot the sum of the series. In MatLab you could define an array of values in $-\pi \leq x \leq \pi$ by:

```
x = -pi + 2*pi*(0:750)/750;
```

Then do a loop to sum (say) the first 3000 terms:

```
f = 0;
for n = 3000:-1:1
    f = f + ((-1)^n)*(sin(n*x)/n);
end
```

Finally, plot:

```
plot(x, f)
```

Look at the plot. Can you get this from your formula?

Note: this series converges rather slowly (this is why I suggest a lot of terms in the summation). This series is also a good example of a **non-uniformly** convergent series of functions. At any given x, the error after summing N terms is less than $C(x)/N$, but $C(x)$ is not bounded.

The stuff below is highly recommended; do it (although we will not grade it, since it is an open-ended thing, what happens here is very important and it will show up later in the course). To see how the convergence actually occurs, try plotting partial sums (as above), but instead of looking at a lot of terms, look at what happens as you add (say) 10, 20, 50, 100, 150 terms. *Look carefully at what goes on near $x = \pi$ and $x = -\pi$.* Now do the same for the series where the n-th term is $\sin(nx)/n$ and look at what happens near $x = 0$.

Solution: To sum the series $f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\sin(nx)}{n}$, we begin by recalling De Moivre's formula:

$$e^{inz} = \cos(nz) + i\sin(nz)$$

and the Taylor expansion for the principal branch of the logarithm about $z = 1$:

$$\mathrm{Log}(1+z) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n \,. \tag{2.1.2}$$

This leads us to consider

$$-\mathrm{Log}(1+e^{ix}) = \sum_{n=1}^{\infty} (-1)^n \frac{e^{inx}}{n} \quad \Longrightarrow \quad \mathrm{Im}\left(-\log(1+e^{ix})\right) = \sum_{n=1}^{\infty} (-1)^n \frac{\sin(nx)}{n}. \tag{2.1.3}$$

Now, since $\mathrm{Log}(1+e^{ix}) = \ln|1+e^{ix}| + i\mathrm{Arg}(1+e^{ix})$, we finally obtain:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx) &= -\mathrm{Arg}(1+e^{ix}) = -\tan^{-1}\left(\frac{\sin(x)}{1+\cos(x)}\right)\\ &= -\tan^{-1}\left(\frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right)\\ &= -\tan^{-1}\left(\frac{\sin(x/2)}{\cos(x/2)}\right)\\ &= -\frac{1}{2}\,x\,.\end{aligned}$$

The last equality here is valid only for $-\pi < x < \pi$ (outside this range different branches of $\tan^{-1}$ come into play). Obviously $\theta = \mathrm{Arg}(1+e^{ix})$ is periodic of period 2π and takes values only in the range $|\theta| \leq \frac{\pi}{2}$ (this all agrees with the fact that the infinite sum on the left should be periodic of period 2π in x). Thus, for example, for $\pi < x < 3\pi$ the sum is equal to $\pi - \frac{1}{2}x$ and for $-3\pi < x < -\pi$ it is equal to $-\pi - \frac{1}{2}x$. **The sum is (in fact) a sawtooth function and (2.1.1) is just its Fourier Series.**

Figures 2.1.1 and 2.1.2 show some partial sums for the series in (2.1.1). Notice that the convergence

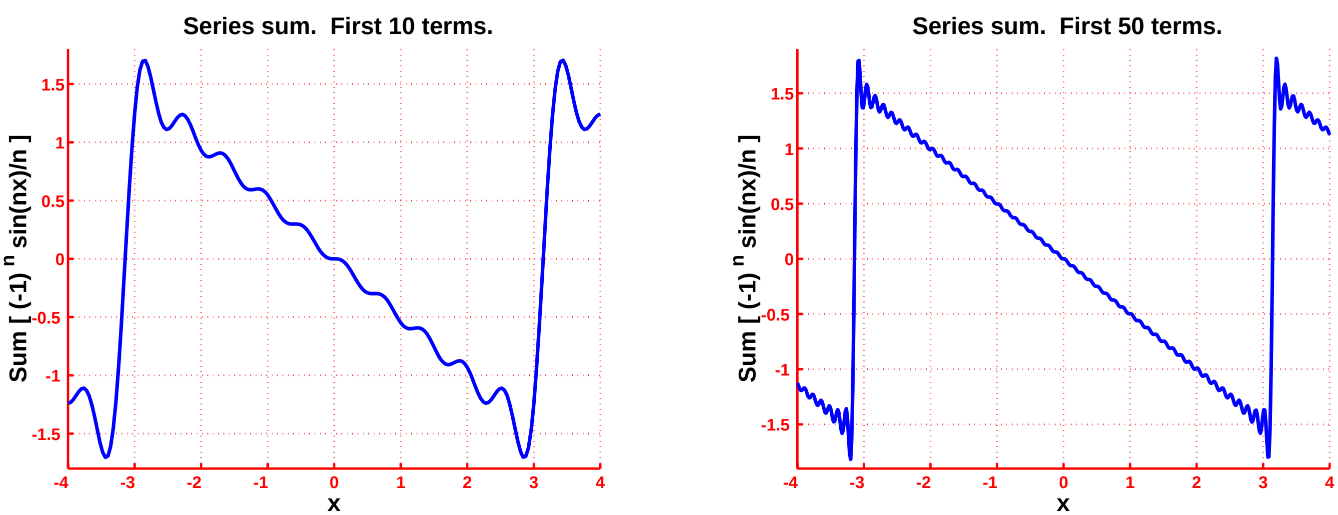

Figure 2.1.1: Plots of the partial sums with 10 and 50 terms for the infinite series in (2.1.1).

to the limit is reasonable for x away from the discontinuities at $x = n\pi$. Near the discontinuities oscillations in the sums appear. These oscillations do not disappear as more terms are added to

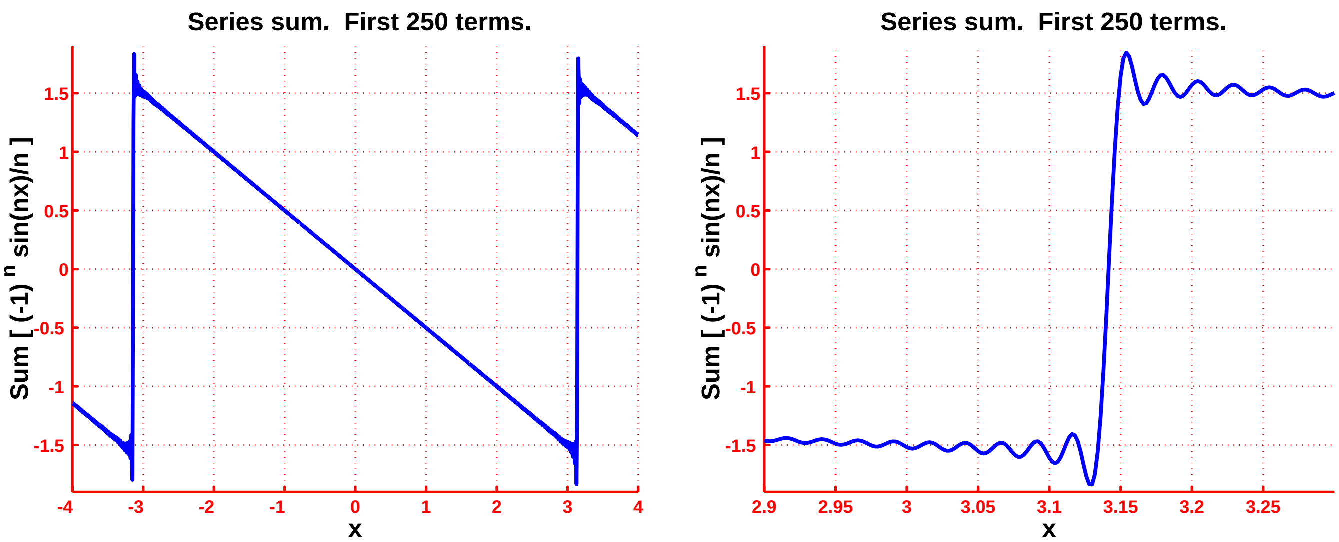

Figure 2.1.2: Plot of the partial sum with 250 terms for the infinite series in (2.1.1), with a blow up of the region near the discontinuity at $x = \pi$.

the sum. In fact, what happens is *(let N be the number of terms added in the partial sum for the series)* that:

- The amplitude of the oscillations remains constant, about 12% of the jump across the discontinuity.
- The region over which the oscillations occur gets narrower, with its width $O(N^{-1})$. This is illustrated by the detail shown on the right picture in figure 2.1.2.

The fact that oscillation appear near discontinuities in the partial sums of Fourier Series (as described above) is general and it is known by the name of

Gibbs Phenomenon.

It is related to the fact that series such as (2.1.1) do not converge uniformly in x. You can experiment more with Fourier Series and the Gibbs phenomenon using the MatLab scripts in the 18.04 Toolkit in Athena (specifically: GibbsDemo and sinSERIES).

Finally: **an important clarifications that we should make regarding the calculations here:** Generally we can only be sure of the convergence of a Taylor series *inside* its circle of convergence.

However, above in (2.1.3) we used the Taylor series (2.1.2) for $\text{Log}(1+z)$ *right on* its circle of convergence! **This needs some justification:**

We show now that *the Taylor series (2.1.2) converges everywhere in its circle of convergence, except for* $z = -1$ (but the convergence is not uniform). We start with the formula for the partial sums of the geometric series

$$\sum_0^N (-1)^n \zeta^n = \frac{1 + (-1)^N \zeta^{N+1}}{1+\zeta}\,.$$

Integrate now both sides in this equality along the path $\zeta = rz$ in the complex ζ–plane, where $0 \leq r \leq 1$ and $z \neq -1$ is some arbitrary point with $|z| \leq 1$. This yields:

$$\left.\begin{aligned} -\sum_1^{N+1} \frac{(-1)^n}{n} z^n &= \int_0^1 \frac{1 + (-1)^N z^{N+1} r^{N+1}}{1+rz} z\,dr \\ &= \int_0^1 \frac{z}{1+rz} dr + (-1)^N z^{N+2} \int_0^1 \frac{r^{N+1}}{1+rz} z\,dr \\ &= \text{Log}(1+z) + (-1)^N z^{N+2} \int_0^1 \frac{r^{N+1}}{1+rz} z\,dr\,. \end{aligned}\right\} \tag{2.1.4}$$

Now, because $|z| \leq 1$ and $z \neq -1$, it follows that: $\boxed{|1+rz| \geq M = M(z) \quad \text{for} \quad 0 \leq r \leq 1,}$ where $\boxed{M(z) > 0}$ (though M vanishes as $z \to -1$). Thus we can estimate the size of the last term in (2.1.4) by:

$$\left| (-1)^N z^{N+2} \int_0^1 \frac{r^{N+1}}{1+rz} z\,dr \right| \leq \frac{1}{M} \int_0^1 r^{N+1} dr = \frac{1}{(N+2)M}\,. \tag{2.1.5}$$

Using this in (2.1.4), we see that the sum on the left converges to $\text{Log}(1+z)$ as $N \to \infty$ — which is what we set out to prove.

Notice that the estimate in (2.1.5) is not uniform in z, since M vanishes for $z \to -1$. Thus the convergence of the Taylor series degrades as we approach $z = -1$.

Remark 2.1.1 *In the problem statement it was mentioned that the series in (2.1.1) did not converge uniformly, with an error on the partial sums of the form $C(x)/N$, where C was not bounded. Applying the results of the proof above to $z = e^{ix}$ and using (2.1.3), we see that this result follows — with* $C = \dfrac{1}{M(e^{ix})}$.

THE END.