Answers to Problem Set Number 5 for 18.04. MIT (Fall 1999)

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November 4, 1999

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1 Problems from the book by Saff and Snider.

1.1 Problem 04 in section 4.4.

Let us first of all parametrise the contour Γ_0:

$$\Gamma_0 : \ z(t) = \begin{cases} e^{it}, & 0 \leq t \leq 2\pi\,, \\ e^{i(4\pi - t)}, & 2\pi \leq t \leq 4\pi\,. \end{cases}$$

We want to deform this contour continuously to the contour $\Gamma_1 : \ z(t) = 1, \ 0 \leq t \leq 4\pi$. This is achieved by the following intermediate set of contours,

$$\Gamma_s : \ z(s,t) = \begin{cases} e^{i(1-s)t}, & 0 \leq t \leq 2\pi\,, \\ e^{i(1-s)(4\pi - t)}, & 2\pi \leq t \leq 4\pi\,, \end{cases}$$

where $0 \leq s \leq 1$.

To check that the conclusion of the *Deformation Invariance Theorem* holds for this example, let us split the contour Γ_0 into two parts: Let γ_a be the part of the contour that has counterclockwise direction and let γ_b be the part of the contour that has clockwise direction. Notice that $\gamma_b = -\gamma_a$, so that (by equation (3) of page 115 in the book):

$$\int_{\Gamma_0} f(z)dz = \int_{\gamma_a} f(z)dz + \int_{\gamma_b} f(z)dz = \int_{\gamma_a} f(z)dz - \int_{\gamma_a} f(z)dz = 0\,.$$

For the integral along Γ_1, we can use theorem 5 of section 4.2:

$$\left| \int_{\Gamma_1} f(z)dz \right| \leq \max_{z \in \gamma_1} |f(z)| \times \textbf{length}(\Gamma_1) = |f(1)| \times 0 = 0\,.$$

Thus both integrals (along Γ_0 and Γ_1) vanish, and therefore are equal.

1.2 Problem 07 in section 4.4.

Part (a)

Since $f(z) = u(x,y) + iv(x,y)$ is analytic, we know that u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

The vector field corresponding to $\bar{f} = u - iv$ is $\mathbf{V} = (u, -v)$, that is: $V_1 = u$ and $V_2 = -v$. Thus,

$$\begin{aligned}\frac{\partial V_1}{\partial y} &= \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial V_2}{\partial x}\,,\\ \frac{\partial V_1}{\partial x} &= \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial V_2}{\partial y}\,.\end{aligned}$$

From the first line here we see that the vector field is irrotational and from the second we see that it is also solenoidal.

Part (b)

Now suppose that $\mathbf{V} = (V_1, V_2)$ is a continuously differentiable, irrotational, solenoidal vector field. Thus

$$\frac{\partial V_1}{\partial y} = \frac{\partial V_2}{\partial x} \quad \text{and} \quad \frac{\partial V_1}{\partial x} = -\frac{\partial V_2}{\partial y}\,.$$

If we now let $f = V_1 - iV_2$, we see that the real and imaginary parts of f satisfy the Cauchy-Riemann equations and are continuously differentiable. By Theorem 5 of section 2.4, f is differentiable and hence analytic.

1.3 Problem 10 in section 4.4.

Consider the contour integral (for the choices of f below)

$$\oint_{|z|=2} f(z)\, dz\,. \tag{1.3.1}$$

Part (a)

$f(z) = \dfrac{z}{z^2+25} = \dfrac{z}{(z+25i)(z-25i)}$. This function is analytic everywhere, except at the values $z = \pm 25i$. Since both of these points lie outside the circle $|z| = 2$, the integral (1.3.1) vanishes by Cauchy's Theorem.

Part (b)

$f(z) = e^{-z}(2z+1)$. This function is analytic everywhere, so that the integral (1.3.1) vanishes by Cauchy's Theorem.

Part (c)

$f(z) = \dfrac{\cos z}{(z-3+i)(z-3-i)}$. This function is analytic everywhere, except at the values $z = 3 \pm i$. Since both of these points lie outside of the circle, the integral (1.3.1) vanishes by Cauchy's Theorem.

Part (d)

$f(z) = \text{Log}(z+3)$. This function is analytic everywhere, except when $\text{Arg}(z+3) = \pi$, i.e. except on the real axis with $\text{Re}(z) < -3$. So it is analytic inside the circle and the integral (1.3.1) vanishes by Cauchy's Theorem.

Part (e)

$f(z) = \sec(z/2) = 1/\cos(z/2)$. This function is analytic everywhere, except when the denominator vanishes, which occurs when $z = \pi + 2k\pi$ for k an integer. Since $-\pi < 2$ and $\pi > 2$, this function is analytic inside the circle and the integral (1.3.1) vanishes by Cauchy's Theorem.

1.4 Problem 18 in section 4.4.

Consider the contour integral

$$I = \oint_{|z|=2} \frac{dz}{z^2(z-1)^3}\,.$$

We show now that this integral vanishes.

Step (a)

The integrand is a function analytic everywhere, except when $z = 0$ or $z = 1$. We can define the domain D to be the complex plane without the interior of the circle $|z| = 1.5$ for example. (Note that this domain is not simply connected, but it doesn't need to be for this argument.) The domain D contains the circles $|z| = 2$ and $|z| = R$ for any $R > 2$, and these circles can be continuously deformed into each other (by continuously varying the radius). We can therefore use theorem 8 of section 4.4, to conclude that $I = I(R)$ for every $R > 2$.

Step (b)

On the contour, we have $z = R\cos\theta + iR\sin\theta$. Thus $|z| = R$ and

$$|z-1| = \sqrt{(R\cos\theta - 1)^2 + R^2\sin^2\theta} = \sqrt{R^2 + 1 - 2R\cos\theta} \geq \sqrt{R^2 + 1 - 2R} = R - 1\,.$$

Thus (on the contour) we have $\left|\dfrac{1}{z^2(z-1)^3}\right| \leq \dfrac{1}{R^2(R-1)^3}$ and by theorem 5 of section 4.2,

$$|I(R)| \leq \frac{1}{R^2(R-1)^3} \times (\textbf{length of path}) = \frac{2\pi}{R(R-1)^3}. \tag{1.4.1}$$

Step (c)

From (1.4.1) it follows that $\lim\limits_{R\to\infty} I(R) = 0$.

Step (d)

We know that $I = I(R)$ for all $R > 2$. Now, suppose $I \neq 0$, say $|I| = \epsilon > 0$. But (from part (c)) we know that there is an $R_0 > 2$, such that $|I(R)| < \epsilon$ for all $R \geq R_0$. But then, for $R \geq R_0$ we have $\epsilon = |I| = |I(R)| < \epsilon$, which is a contradiction. Hence $I = 0$.

1.5 Problem 06 in section 4.5.

Consider the integral

$$I = \int_\Gamma \frac{e^{iz}}{(z^2+1)^2}dz = \int_\Gamma \frac{e^{iz}}{(z+i)^2(z-i)^2}dz\,, \tag{1.5.1}$$

where Γ is the circle $|z| = 3$ traversed once in the counterclockwise direction. Since the integrand is analytic everywhere, except at $z = \pm i$, we can use the deformation invariance theorem (theorem 8 of section 4.4) and apply it to the domain D, where D is the complex plane without the points $\pm i$.

We deform the contour as indicated in figure 1.5.1, to obtain

$$\begin{aligned} I &= \int_{\gamma_1} \frac{e^{iz}}{(z^2+1)^2}dz + \int_{\gamma_2} \frac{e^{iz}}{(z^2+1)^2}dz \\ &= 2\pi i \left[\frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\bigg|_{z=i} + \frac{d}{dz}\left(\frac{e^{iz}}{(z-i)^2}\right)\bigg|_{z=-i}\right] \\ &= \frac{\pi}{e}\,. \end{aligned}$$

where (in the second step) we used theorem 19 of section 4.5.

1.6 Problem 08 in section 4.5.

Consider the circle $|z - z_0| = r$. Let us parametrise it as follows: $z(\theta) = z_0 + re^{i\theta}$, $0 \leq \theta \leq 2\pi$. Then Cauchy's Integral Formula tells us that if f is analytic inside and on the circle, then

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\oint_{|z-z_0|=r} \frac{f(z)}{z-z_0}dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{z_0+re^{i\theta}-z_0} rie^{i\theta}d\theta \end{aligned}$$

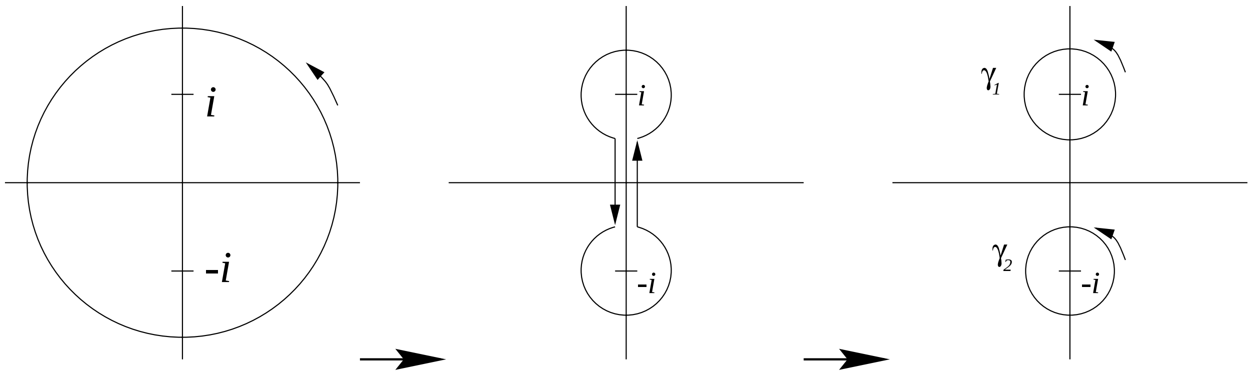

Figure 1.5.1: Deformation of the contour Γ in (1.5.1).

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta\,,$$

where we have used the fact that $\dfrac{dz}{d\theta} = rie^{i\theta}d\theta$. More generally, using theorem 19 of section 4.5:

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i}\oint_{|z-z_0|=r}\frac{f(z)}{(z-z_0)^{n+1}}dz \\ &= \frac{n!}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}}rie^{i\theta}d\theta \\ &= \frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0+re^{i\theta})e^{-in\theta}d\theta\,. \end{aligned}$$

1.7 Problem 14 in section 4.5.

Consider the function

$$G(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\cos\zeta}{\zeta - z}d\zeta\,,$$

where Γ is a simple closed positively oriented contour that passes through the point $2+3i$. The function $\cos(\zeta)$ is is analytic everywhere, so that $\dfrac{\cos(\zeta)}{(\zeta - z)}$ is analytic everywhere, except when $\zeta = z$.

- If z lies outside of Γ, then from Cauchy's Theorem $G(z) = 0$, since the integrand is analytic inside the contour. Thus

$$\boxed{\lim_{z\to 2+3i} G(z) = 0 \;\; \textbf{when } z \textbf{ approaches } 2+3i \textbf{ from outside } \Gamma.}$$

- If z lies inside Γ, we can use Cauchy's Integral Formula to evaluate the integral. Then $G(z) = \cos(z)$. Thus

$$\boxed{\lim_{z \to 2+3i} G(z) = \cos(2+3i) \ \textbf{ when } z \textbf{ approaches } 2+3i \textbf{ from inside } \Gamma\textbf{.}}$$

1.8 Problem 06 in section 4.6.

Let $f(z)$ be an entire function such that $f^{(5)}(z)$ is bounded in the whole plane. Then f is a polynomial of degree at most 5.

Since f is entire, it is infinitely many times differentiable, and each of its derivatives is also entire. In particular, $f^{(5)}$ is entire and (by assumption) it is bounded in the whole plane. By theorem 21 of section 4.6 **(Liouville's Theorem)**, we conclude that $f^{(5)}$ is a constant function. Integrating $f^{(5)}$ five times we see that f must be a polynomial of degree at most 5.

1.9 Problem 14 in section 4.6.

Minimum Modulus Principle: *Let f be analytic in a bounded domain D and continuous up to and including its boundary. Then, if f is* non-zero *in D, the modulus $|f(z)|$ attains its minimum value on the boundary of D.*

To show this we will apply the maximum modulus principle to the function $g(z) = 1/f(z)$. However, we must be careful, since the maximum modulus principle requires that the function it is applied to be analytic in the domain D and continuous up to and including its boundary. It should be clear that the only way $g(z)$ can fail to satisfy these conditions is if $f(z)$ vanishes somewhere (which, by hypothesis, can only happen on the boundary). **Thus we distinguish two cases:**

- **i)** $f(z_0) = 0$ at some point z_0 on the boundary. Then, since $f(z) \neq 0$ in D, $|f(z)| > 0$ in D and (by continuity) $|f(z)| \geq 0$ on the boundary of D. Since $|f(z_0)| = 0$, $|f(z)|$ does indeed attain its minimum on the boundary.

- **ii)** On the other hand, assume that $f(z) \neq 0$ in D and on the boundary. Then $g = g(z)$ satisfies the conditions for the maximum modulus principle, so that $|g(z)|$ attains its maximum on the boundary. But a maximum of $|g(z)|$ is a minimum of $|f(z)|$. Thus, again, $|f(z)|$ attains its minimum on the boundary.

Counterexample 1: Consider $f(z) = z$ on the unit disk $|z| < 1$, which satisfies the conditions for the Minimum Modulus Principle, *except that* $f(0) = 0$. In this case $|f(z)| = 1$ on the boundary of the domain (the unit circle), while the minimum of $|f|$ is clearly 0. Thus, **if the condition that f be non-zero on D fails, the Minimum Modulus Principle need not apply.**

Counterexample 2: Consider $f(z) = e^{-z}$ on the right hand side of the complex plane $\mathrm{Re}(z) > 0$, which satisfies the conditions for the Minimum Modulus Principle, *except that the domain is not bounded.* In this case $|f(z)| = 1$ on the boundary of the domain (the imaginary axis), while the minimum of $|f|$ is clearly 0 (look at the values of $|f(z)|$ on the positive real axis). Thus, **if the condition that the domain be bounded fails, the Minimum Modulus Principle need not apply.**

1.10 Problem 24 in section 4.6.

Here we show that **if P is a polynomial that has no zeros on a simple positively oriented contour Γ,** then

$$I = \frac{1}{2\pi i} \oint_\Gamma \frac{P'(z)}{P(z)} dz \tag{1.10.1}$$

gives the number of zeros (counting multiplicity) that P has inside the contour Γ.

We can write $P(z) = c \prod_{k=1}^{n} (z - z_k)$, where the z_k's are the zeros of $P(z)$ (occurring with their multiplicities) and c is some constant. Then, using the product rule to differentiate $P(z)$, we find

$$\frac{dP}{dz} = c \sum_{\ell=1}^{n} \prod_{k=1\,\&\,k\neq\ell}^{n} (z - z_k)\,,$$

so that

$$\frac{P'(z)}{P(z)} = \sum_{\ell=1}^{n} \frac{1}{z - z_\ell}\,.$$

We recall now that

$$\frac{1}{2\pi i} \oint_\Gamma \frac{1}{z - z_\ell} dz = \begin{cases} 1 & \text{if } z_\ell \text{ is inside } \Gamma\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$\frac{1}{2\pi i} \oint_\Gamma \frac{P'(z)}{P(z)} dz = \sum_{\ell=1}^{n} \left(\frac{1}{2\pi i} \oint_\Gamma \frac{1}{z - z_\ell} dz \right) = \text{No. of zeros of P inside } \Gamma \text{, counting multiplicities.}$$

THE END.