Answers to Problem Set Number 4 for 18.04. MIT (Fall 1999)

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1 Problems from the book by Saff and Snider.

1.1 Problem 10 in section 4.2.

First notice that the path C is not a smooth curve but consists of four smooth curves (the four sides of the square). Let us call the four sides $C_1, \ldots, C_4$, starting with $C_1 = [0, 1]$ and proceeding in counterclockwise order.

According to definition 4 in section 4.2 we then have

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4},$$

which reduces the problem to the calculation of integrals over four smooth parts. To compute the individual integrals, we need to find a parameterization for each C_i first. A natural choice is given by

$$\begin{aligned} C_1 &: \; z_1(t) = t\,, \\ C_2 &: \; z_2(t) = 1 + it\,, \\ C_3 &: \; z_3(t) = 1 + i - t\,, \\ C_4 &: \; z_4(t) = i - it\,, \end{aligned}$$

where, in all cases: $0 \leq t \leq 1$. Then, for each side $\boxed{\int_{C_i} \bar{z}^2 dz = \int_0^1 z_i\bar{(t)}^2 z_i'(t) dt,}$ so that

$$\begin{aligned} \int_{C_1} \bar{z}^2 dz &= \int_0^1 t^2 dt\,, \\ \int_{C_2} \bar{z}^2 dz &= \int_0^1 (1 - it)^2 i dt\,, \\ \int_{C_3} \bar{z}^3 dz &= \int_0^1 (1 - i - t)^2 (-1) dt\,, \\ \int_{C_4} \bar{z}^3 dz &= \int_0^1 (1 - it)^2 (-i) dt\,. \end{aligned}$$

Adding everything up together, we get

$$\begin{aligned} \int_C \bar{z}^2 dz &= \int_0^1 \left(t^2 + i(1 - it)^2 - (1 - t - i)^2 - i(i - it)^2\right) dt \\ &= \int_0^1 (4t + i(4 - 4t))\, dt \\ &= \left(2t^2 + i(4t - 2t^2)\right)\Big|_0^1 \\ &= 2 + 2i\,. \end{aligned}$$

1.2 Problem 14b in section 4.2.

On the contour γ, we can write $z = R + iy$, where $0 \leq y \leq 2\pi$. We now look for an upper bound on $|f(z)|$ for $z \in \gamma$. We have the identity:

$$|f(z)| = \left| \frac{e^{3R+3it}}{1+e^{R+it}} \right| = \frac{|e^{3R+3it}|}{|1+e^{R+it}|} \,.$$

Using now the fact that $|e^{3it}| = 1$ and the inequality

$$|1+e^{R+it}| \geq |e^{R+it}| - 1 = e^R - 1 > 0 \,,$$

we obtain

$$|f(z)| = \frac{|e^{3R+3it}|}{|1+e^{R+it}|} \leq \frac{e^{3R}}{e^R - 1} \quad \text{for} \quad z \in \gamma \,.$$

We now use this bound and theorem 5 in section 4.2 for the integral of f over γ to obtain:

$$\left| \int_\gamma f(z) dz \right| \leq \frac{e^{3R}}{e^R - 1} \textbf{length}(\gamma) = \frac{2\pi e^{3R}}{e^R - 1} \,.$$

1.3 Problem 2 in section 4.3.

Any polynomial $P = P(z)$ can be written in the form

$$P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n \,,$$

where n is a non-negative integer and the a_p's are complex numbers. Each of the monomials z^p $(0 \leq p \leq n)$ has an antiderivative, namely: $\dfrac{1}{p+1} z^{p+1}$, which is an entire function. Because P is a linear combination of these monomials, P also has an antiderivative:

$$Q(z) = \frac{a_0}{n+1} z^{n+1} + \frac{a_1}{n} z^n + \ldots + a_n z \,.$$

From theorem 7 in section 4.3, every complex function with an antiderivative in a domain has a vanishing integral over any closed loop within this domain (in this case the domain can be taken as the whole complex plane).

1.4 Problem 7 in section 4.3.

Computing the integral directly using a parameterization is quite complicated in this problem. Instead, we will to show that the function $f = f(z) = \dfrac{1}{z - z_0}$ has an antiderivative that is defined on a domain containing C.

The natural candidates are branches of the multiple valued function $\log(z - z_0)$. This function has branch points at $z = z_0$ and at $z = \infty$ (nowhere else) which we need to avoid. Since z_0 is outside C, we can always find a domain D that contains C but not z_0. For example an open disk with the same center as C and a slightly bigger radius, but not big enough to include z_0. Then we can find a branch cut for $\log(z - z_0)$ (a curve joining z_0 with ∞) which does not intercept D and use it to define a branch for the $\log(z - z_0)$ function, which then is an antiderivative for $\dfrac{1}{z - z_0}$.

Once we find such an antiderivative in the domain D, we conclude that the integral of $f = f(z)$ over any closed contour in D (this obviously includes C as a particular case) is zero.

Remark 1.4.1 *We can also start with a branch cut that does not intercept C, and then choose D to be the complement of this branch cut (i.e. the whole complex plane minus the branch cut).*

2 Other problems.

2.1 Problem 4.1 in 1999.

Statement: Find all the branch points of the following function

$$f(z) = \log(1 - \sqrt{z}) \tag{2.1.1}$$

and choose branch cuts to make it single-valued.

Solution: Let $g(z) = 1 - \sqrt{z}$. Then the **possible** branch points of the function $\log(g(z))$ are:

- The branch points of $g(z)$, which are $z = 0$ and $z = \infty$.
- The points where $g(z)$ takes as a value a branch point of the log function. That is, the solutions of the equations: $g(z) = 1$ and $g(z) = \infty$. These yield $z = 1$ and $z = \infty$.

It is not guaranteed that these candidates are indeed the branch points. We must study the behavior of $\log(1 - \sqrt{z})$ near them and check whether they actually are branch points or not. We do this next.

- **Point $z = 0$.** Consider a small circle around $z = 0$. We start at a point **P** on this circle and travel counter-clockwise. Since $z = 0$ is a branch point of $\sqrt{z}$, the value of $\sqrt{z}$ does not return to its initial value when we return to **P**, but to its negative.

  Because any single-valued branch of the log is a one-to-one function,[1] the value of $\log(1 - \sqrt{z})$ will also return to a different value as we return to **P**. Therefore:

  $$\boxed{z = 0 \text{ is a branch point of } f.}$$

  To see the "geometry" behind the argument above,[2] note that: As z traces a small circle around zero, $\sqrt{z}$ traces **one half** of a circle around zero. Thus, $g = g(z)$ (as defined above) traces one half of a (small) circle around $g = 1$. But the logarithm is analytic at one, so for values of its argument near one we have $\log(g) = \log(1) + (g - 1) + \epsilon(g - 1)$, where $|\epsilon(g - 1)|$ is much

[1] To see this, note that if $\log(z_1) = \log(z_2) = \zeta$, then $z_1 = z_2 = e^{\zeta}$.

[2] It is always useful to do this kind of visualization, as it help in the understanding of how branch points "work".

smaller than $|g-1|$ when $|g-1|$ is small.[3] *It follows that, as z traces a small circle around $z=0$, $f(z)$ above in (2.1.1) traces (approximately)* **one half of a small circle** *around* $\log(1)$. *Hence $z=0$ is a branch point for f.*

- **Point $z=\infty$.** Now we use the fact that (for $|z|$ large) $\log(1-\sqrt{z})$ behaves like $\log(-\sqrt{z}) = \frac{1}{2}\log(z)+i\pi$, which has a branch point at ∞. Therefore:

$$\boxed{z=\infty \textbf{ is a branch point of } f\textbf{.}}$$

- **Point $z=1$.** This one is subtle and the answer will very much depend on which branch of the square root we are at. Namely:

  - **Case** $\sqrt{1}=-1$. Then $g(1)=2$ and $z=1$ is not a branch point of f in (2.1.1), since $g=g(z)$ is analytic at $z=1$ and $\log(g)$ is analytic at $g=2$. Thus, the argument that lead us to pick $z=1$ as a possible branch point (right at the beginning of this solution) does not apply.

  - **Case** $\sqrt{1}=1$. Then $g(1)=0$ and we do "hit" a branch point of logarithm. Thus, consider what happens as a small circular path centered at $z=1$ is traced: because g is analytic at one, $g(z)$ will then (approximately) trace a small circle around $g=0$ (the argument here is entirely similar to the one we used above when visualizing the geometry of the case $z=0$). But then $\log(g)$ will jump by $2\pi i$ (zero is a branch point for the logarithm). It follows then that in this case $z=1$ is a branch point.

We conclude that

$$\boxed{\begin{array}{l} \textbf{For } \sqrt{1} = \ \ 1 \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ z=1 \textbf{ is a branch point of } f\textbf{,} \\ \textbf{For } \sqrt{1} = -1 \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ z=1 \textbf{ is not a branch point of } f\textbf{.} \end{array}}$$

That is, in terms of the "Riemann Surface" (the surface made by "gluing" appropriately all the branches of f in (2.1.1)), $z=1$ is a branch point in some levels an not in others (see remark 2.1.1 below).

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[3] This is just the definition of having a derivative.

How do we choose the branch cuts? Well, this would depend on which sheet we are in. If $\sqrt{(1)} = -1$, then a suitable branch cut would be any curve joining $z = 0$ and $z = \infty$. On the other hand, if $\sqrt{(1)} = 1$, a suitable branch cut would be any curve going from one branch point to the next to the next (in any order). A few **examples in this second case are:**

- Use the positive real axis.
- Use the lines defined by $\mathrm{Im}(z) \geq 0$ and $\mathrm{Re}z = 0$ or $\mathrm{Re}z = 1$.
- Use the real axis for $\mathrm{Re}z \leq 0$ and $\mathrm{Re}z \geq 1$.

Remark 2.1.1 *Concerning the situation with the point $z = 1$. Recall the analogy made in the* Branch Points Handout *(and in the lectures) of the Riemann Surface (for some examples) with some sort of "car parking building". In this context, notice that the branch points are the points "organizing" the ramps that allow the cars to change levels (the ramps wind around the branch points). The fact that $z = 1$ is a branch point in some places and not in others simply means that: there are some levels where you can find a ramp (taking you to a different level) that goes around $z = 1$ and there are other levels where this does not happen. This is not that strange (though it may cause some confusion to the poor drivers trying to get out of there).*

Visualizing the Riemann Surface for the function in (2.1.1) is not easy, because the surface cannot be constructed as a surface within three dimensional space (this is due to the effect of the square root, that gets back to its starting value after two turns around its branch point ... very much as in an Escher picture!). Thus we will not attempt to produce a plot of the whole Riemann Surface here. Instead we will partition the surface into appropriate Riemann Sheets, which we can then use as Lego blocks to construct the surface. *These are shown in figures 2.1.1 and 2.1.2.*

Figure 2.1.1 shows a sheet of the Riemann Surface where $z = 1$ is a branch point and figure 2.1.2 shows a sheet where $z = 1$ is not a branch point. We can construct the whole surface using these two pieces as Lego blocks in the following fashion:

- *Take infinite copies of the sheet in figure 2.1.1 and stack them one on top of the other, joining the matching lips of the branch cut along the real axis on $x \geq 1$. These are marked in the figure by the thick lines colored red (lower lip) and blue (upper lip). The matching is blue in any copy to red in the copy immediately above.*

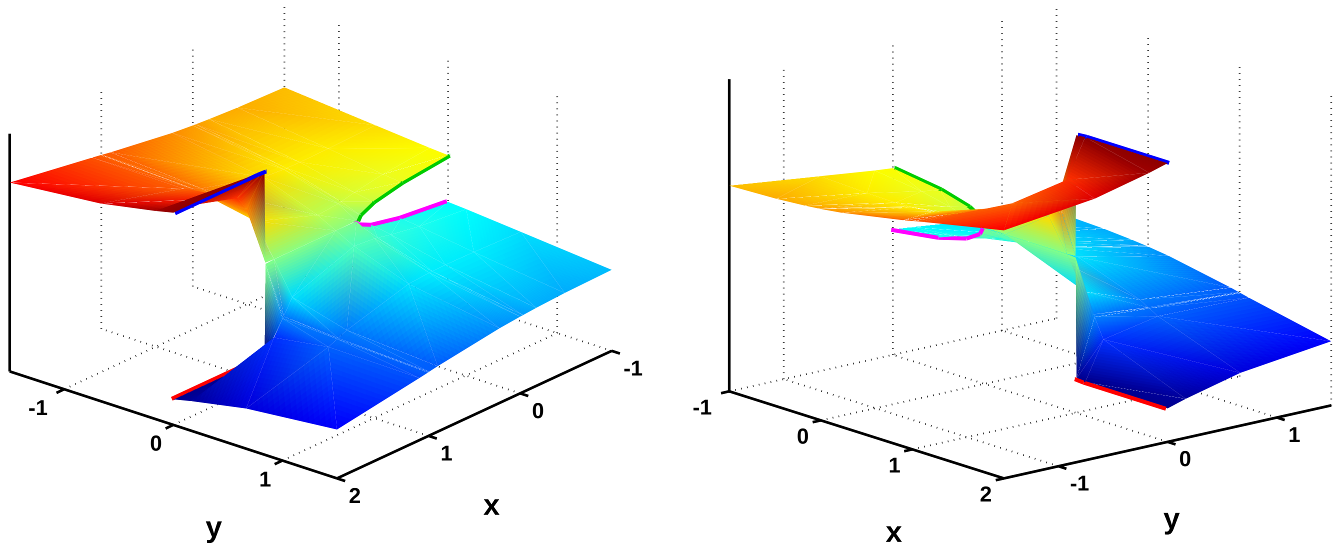

Figure 2.1.1: Two views of a sheet of the Riemann Surface for (2.1.1) where $z = 1$ is a branch point.

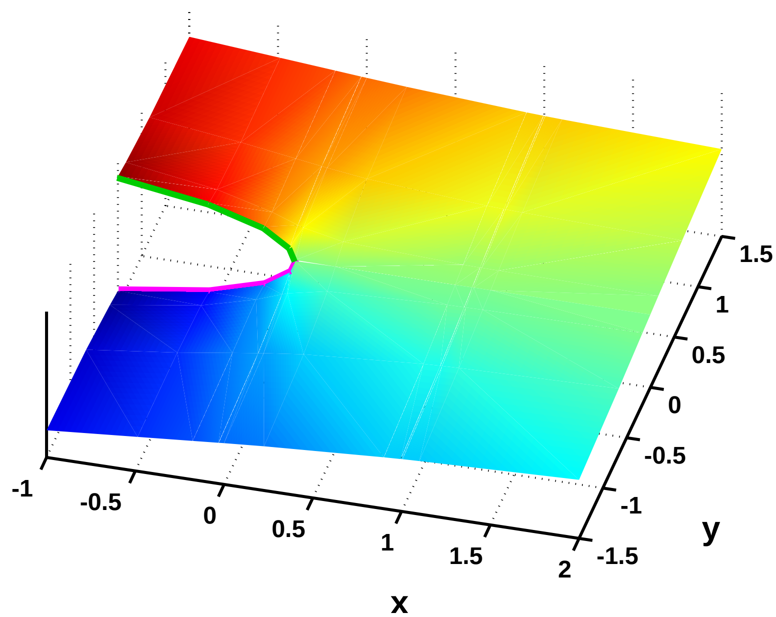

Figure 2.1.2: Sheet of the Riemann Surface for (2.1.1) where $z = 1$ is not a branch point.

- *The first step leaves us with a surface that is still contained in three dimensional space. However, we have an infinite set of gaps on it; in each level there is an opening along the negative real axis (on $x \leq 0$) due to the other branch cut. As a construction company, we would be in* *serious trouble if any car fell off through one of this gaps.*

- *If we were constrained to have our parking building within the normal three dimensional space, we would be in serious trouble trying to get rid of the gaps left in the prior step. But mathematicians have contacts in high places, so we will be allowed to spill our construction into the fourth dimension (which we do next).*

- *Take now infinite copies of the sheet in figure 2.1.2. Notice that this sheet has only one branch cut, with its lips marked also by thick lines, colored green (upper lip) and magenta (lower lip) — same scheme used to color the lips of the $x \leq 0$ branch cut in the other figure. Now we can close the gaps in our unfinished surface by joining these new Lego blocks along the branch cut lips (matching colors). We can only pull this last step by moving into higher dimensions, so that an intersection of the two types of Lego blocks can be avoided.*

- *That is it, the surface is finished and ready to be put to use.*

2.2 Problem 4.2 in 1999.

Statement: Find all the branch points of the following function

$$f(z) = \log(\log(z)) \tag{2.2.1}$$

and choose branch cuts to make it single-valued.

Solution: Let $g(z) = \log(z)$. Then the **possible** branch points of the function $\log(g(z))$ are:

- The branch points of $g(z)$, which are $z = 0$ and $z = \infty$.

- The points where $g(z)$ takes as a value a branch point of the log function. That is, the solutions of the equations: $g(z) = 0$ and $g(z) = \infty$. These yield $z = 1$, $z = 0$ and $z = \infty$.

It is not guaranteed that these candidates are indeed the branch points. We must study the behavior of $f(z) = \log(\log(z))$ near them and check whether they actually are branch points or not. We do this next.

- **Point** $z = 0$**.** Consider a small circle around $z = 0$. We start at a point **P** on this circle and travel counter-clockwise. When we return to **P** the value of $g(z)$ will have increased by $2\pi i$.

  Because any single-valued branch of the log is a one-to-one function,[4] the value of $\log(\log(z)) = \log(g(z))$ will also return to a different value as we return to **P**. Therefore:

$$\boxed{z = 0 \textbf{ is a branch point of } f\textbf{.}}$$

- **Point** $z = \infty$**.** Let $w = \dfrac{1}{z}$, then:

$$\log(\log(z)) = \log\left(\log\left(\frac{1}{w}\right)\right) = \log(-\log(w)) = \pi i + \log(\log(w)) = \pi i + f(w)\,.$$

  Using now the prior result ($z = 0$ is a branch point for $f(z)$) we see that

$$\boxed{z = \infty \textbf{ is a branch point of } f\textbf{.}}$$

- **Point** $z = 1$**.** This one is subtle and the answer will very much depend on which branch of the logarithm we are at. Namely:

  - **Case** $\log(1) = g(1) = 2n\pi i$, with $n \neq 0$ an integer. Then $z = 1$ is not a branch point of f in (2.2.1), since $g = g(z)$ is analytic at $z = 1$ and $\log(g)$ is analytic at $g = g(1)$. Thus, the argument that lead us to pick $z = 1$ as a possible branch point does not apply.

  - **Case** $\log(1) = g(1) = 0$. Then we do "hit" a branch point of logarithm. Thus, consider what happens as a small circular path centered at $z = 1$ is traced: because $g(z)$ is analytic (with a non-zero derivative) at $z = 1$, $g(z)$ will (approximately) trace a small circle[5] around $g = 0$. But then $\log(g)$ will jump by $2\pi i$ (zero is a branch point for the logarithm). It follows then that in this case $z = 1$ is a branch point.

  We conclude that

$$\boxed{\begin{array}{l} \textbf{For } \log(1) = 0 \;\ldots\ldots\ldots\ldots\ldots\; z = 1 \textbf{ is a branch point of } f\textbf{,} \\ \textbf{For } \log(1) \neq 0 \;\ldots\ldots\ldots\ldots\ldots\; z = 1 \textbf{ is not a branch point of } f\textbf{.} \end{array}}$$

[4] To see this, note that if $\log(z_1) = \log(z_2) = \zeta$, then $z_1 = z_2 = e^{\zeta}$.

[5] The argument here is the same that we have used elsewhere: near $z = 1$ we can write (since $g(1) = 0$ and $\dfrac{dg}{dz}(1) = 1$) $\boxed{g(z) = (z-1) + \epsilon(z-1)\,,}$ where $\epsilon(z-1)$ is much smaller than $(z-1)$ when $(z-1)$ is small.

That is, in terms of the "Riemann Surface" (the surface made by "gluing" appropriately all the branches of f in (2.2.1)), $z = 1$ is a branch point in some levels an not in others. The situation is somewhat similar to the one we found while solving the *Other Problem 4.2 in 1999.* The Riemann surface in the current case, however, is much more of a "nightmare" ... can you figure out a clever way to visualize it?

How do we choose the branch cuts? Well, this would depend on which sheet we are in. If $log(1) \neq 0$, then a suitable branch cut would be any curve joining $z = 0$ and $z = \infty$. On the other hand, if $\log(1) = 0$, a suitable branch cut would be any curve going from one branch point to the next to the next (in any order). A few **examples in this second case are:**

- Use the positive real axis.
- Use the lines defined by $\mathrm{Im}(z) \geq 0$ and $\mathrm{Re}z = 0$ or $\mathrm{Re}z = 1$.
- Use the real axis for $\mathrm{Re}z \leq 0$ and $\mathrm{Re}z \geq 1$.

THE END.