Matrix exponentials

Solution suggestions

1. *In this problem,* $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$ *and we are interested in the equation* $\dot{\mathbf{u}} = A\mathbf{u}$.

(a) Find a fundamental matrix $\Phi(t)$ *for* A.

The eigenvalues of A are solutions to $(1 - \lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 = 0$, which are in this case the complex conjugates $1 \pm 2i$.

Choose one of them, say $\lambda_1 = 1 + 2i$, and find an eigenvector v_1 for it.

The eigenvectors corresponding to λ_1 are the vectors in the killed by $A - \lambda_1 I = \begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix}$, so we can take, say, $v_1 = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$.

This means this equation has a complex-valued solution

$$\mathbf{u}(t) = e^{\lambda_1 t} v_1 = e^{(1+2i)t} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$= e^t \begin{bmatrix} \cos 2t + i \sin 2t \\ 2i \cos 2t - 2 \sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} e^t \cos 2t \\ -2e^t \sin 2t \end{bmatrix} + i \begin{bmatrix} e^t \sin 2t \\ 2e^t \cos 2t \end{bmatrix}$$

By taking real and imaginary parts, we get two independent real solutions, and so can read off a fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ -2e^t \sin 2t & 2e^t \cos 2t \end{bmatrix}.$$

The fundamental matrix is not unique, and there are other correct solutions. However, you can check other solutions by observing that any other fundamental matrix $\bar{\Phi}(t)$ should be related to this one by matching up initial conditions; i.e., it should satisfy the equation

$$\bar{\Phi}(t) = \Phi(t)\Phi(0)^{-1}\bar{\Phi}(0).$$

(b) *Find the exponential matrix* e^{At}.

The exponential matrix is the fundamental matrix uniquely determined by the identity initial conditions $e^{A0} = I$.

So we can compute the exponential matrix for this system from any fundamental matrix $\Phi(t)$ found in the first part by using the formula

$$e^{At} = \Phi(t)\Phi(0)^{-1}.$$

For us, $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$, and

$$e^{At} = \begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ -2e^t \sin 2t & 2e^t \cos 2t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^t \cos 2t & \frac{1}{2}e^t \sin 2t \\ -2e^t \sin 2t & e^t \cos 2t \end{bmatrix}.$$

(c) *Find the solution to* $\dot{\mathbf{u}} = A\mathbf{u}$ *with* $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The general strategy is to find the solution with a given initial condition directly from the exponential matrix:

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0) = \begin{bmatrix} e^t \cos 2t & \frac{1}{2}e^t \sin 2t \\ -2e^t \sin 2t & e^t \cos 2t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t(\cos 2t + \sin 2t) \\ 2e^t(\cos 2t - \sin 2t) \end{bmatrix}.$$

In our case, we could have also observed that this initial condition was the sum of the real and imaginary parts of the complex solution we found in part **(a)** and gotten the answer by adding the columns of our fundamental matrix $\Phi(t)$.

(d) *Find a solution to* $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. *What is the general solution? What is the solution with* $\mathbf{u}(0) = \mathbf{0}$?

We guess a constant solution

$$\mathbf{u} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Substituting this into the DE gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

This implies

$$\mathbf{u} = -A^{-1} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$

(You can check that this is indeed a solution by verifying that it satisfies the equation $A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 0$.)

Since all homogeneous solutions are of the form $e^{At} \begin{bmatrix} a \\ b \end{bmatrix}$, the general solution is then given by

$$e^{At} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} e^t(a \cos 2t + \frac{b}{2} \sin 2t) + 1 \\ e^t(b \cos 2t - 2a \sin 2t) - 6 \end{bmatrix},$$

for some constants a, b.

To find the particular solution with $\mathbf{u}(0) = \mathbf{0}$, we plug $t = 0$ into this expression, and get that

$$\begin{bmatrix} a+1 \\ b-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so the desired solution is given by the constants $a = -1$ and $b = 6$:

$$\begin{bmatrix} e^t(-\cos 2t + 3\sin 2t) + 1 \\ e^t(6\cos 2t + 2\sin 2t) - 6 \end{bmatrix}.$$

2. *Suppose* $\mathbf{u_1}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *(the constant trajectory) and* $\mathbf{u_2}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ *are solutions to the equation* $\dot{\mathbf{u}} = B\mathbf{u}$ *for some matrix B.*

(a) *What is the general solution? What is the solution* $\mathbf{u}(t)$ *with* $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? *What is the solution with* $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

The general solution is

$$\mathbf{u}(t) = a\mathbf{u_1}(t) + b\mathbf{u_2}(t) = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} e^t \\ -e^t \end{bmatrix}.$$

We can find the particular solution that satisfies any given initial condition $\mathbf{u}(0)$ explicitly by solving the system of equations given by $a\mathbf{u_1}(0) + b\mathbf{u_2}(0) = \mathbf{u}(0)$ for a and b.

Here $\mathbf{u}_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so the solution with initial condition $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is the constant trajectory

$$2\mathbf{u_1}(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

and the solution with initial condition $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$\frac{1}{2}\mathbf{u}_1(t) + \frac{1}{2}\mathbf{u}_2(t) = \begin{bmatrix} e^t/2 + 1/2 \\ -e^t/2 + 1/2 \end{bmatrix}.$$

(b) *Find a fundamental matrix, and compute the exponential* e^{Bt}. *What is* e^B?

Since the two given solutions are linearly independent, a fundamental matrix is $\begin{bmatrix} 1 & e^t \\ 1 & -e^{-t} \end{bmatrix}.$

3

Evaluating this matrix at $t = 0$ gives $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ which has inverse $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$, so the exponential matrix is

$$e^{Bt} = \begin{bmatrix} 1 & e^t \\ 1 & -e^t \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} e^t + 1 & 1 - e^t \\ 1 - e^t & e^t + 1 \end{bmatrix}$$

Evaluating e^{Bt} at $t = 1$ gives us $e^B = \frac{1}{2} \begin{bmatrix} e + 1 & 1 - e \\ 1 - e & e + 1 \end{bmatrix}$.

(c) *What are the eigenvalues and eigenvectors of B?*

The eigenvalues of e^B are 1 and e, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively.

The eigenvalues of B are the logarithms of the eigenvalues of e^B, so they are 0 and 1, with the same eigenvectors.

We could have also deduced this from the the two solutions given in the problem statement.

(d) *What is B?*

We can find B by using the answer to part **(c)**.

From the equation for the first eigenvalue, $B \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so B has the form $\begin{bmatrix} a & -a \\ c & -c \end{bmatrix}$.

From the equation for the second eigenvalue, $\begin{bmatrix} a & -a \\ c & -c \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so $a = 1/2$ and $c = -1/2$. Thus,

$$B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

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