Definition of Poles

1. Rational Functions

A **rational function** is a ratio of polynomials $q(s)/p(s)$.

Examples. The following are all rational functions. $(s^2+1)/(s^3+3s+1)$, $1/(ms^2+bs+k)$, $s^2+1+(s^2+1)/1$.

If the numerator $q(s)$ and the denominator $p(s)$ have no roots in common, then the rational function $q(s)/p(s)$ is in **reduced form**

Example. The three functions in the example above are all in reduced form.

Example. $(s-2)/(s^2-4)$ is not in reduced form, because $s=2$ is a root of both numerator and denominator. We can rewrite this in reduced form as

$$\frac{s-2}{s^2-4} = \frac{s-2}{(s-2)(s+2)} = \frac{1}{s+2}.$$

2. Poles

For a rational function in reduced form the **poles** are the values of s where the denominator is equal to zero; or, in other words, the points where the rational function is not defined. We allow the poles to be complex numbers here.

Examples. a) The function $1/(s^2+8s+7)$ has poles at $s=-1$ and $s=-7$.
b) The function $(s-2)/(s^2-4) = 1/(s+2)$ has only one pole, $s=-2$.
c) The function $1/(s^2+4)$ has poles at $s=\pm 2i$.
d) The function s^2+1 has no poles.
e) The function $1/(s^2+8s+7)(s^2+4)$ has poles at -1, -7, $\pm 2i$. (Notice that this function is the product of the functions in (a) and (c) and that its poles are the union of poles from (a) and (c).)

Remark. For ODE's with system function of the form $1/p(s)$, the poles are just the roots of $p(s)$. These are the familiar characteristic roots, which are important as we have seen.

3. Graphs Near Poles

We start by considering the function $F_1(s) = \frac{1}{s}$. This is well defined for every complex s except $s=0$. To visualize $F_1(s)$ we might try to graph it. However it will be simpler, and yet still show everything we need, if we graph $|F_1(s)|$ instead.

To start really simply, let's just graph $|F_1(s)| = \frac{1}{|s|}$ for s real (rather than complex).

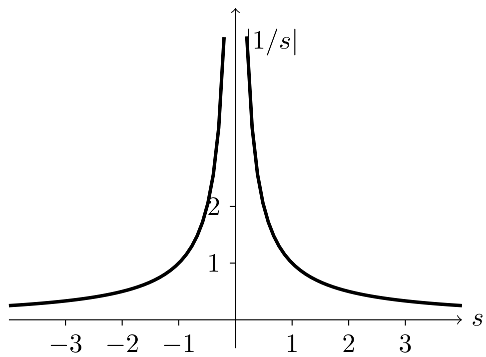

Figure 1: Graph of $\frac{1}{|s|}$ for s real.

Now let's do the same thing for $F_2(s) = 1/(s^2 - 4)$. The roots of the denominator are $s = \pm 2$, so the graph of $|F_2(s)| = \frac{1}{|s^2-4|}$ has vertical asymptotes at $s = \pm 2$.

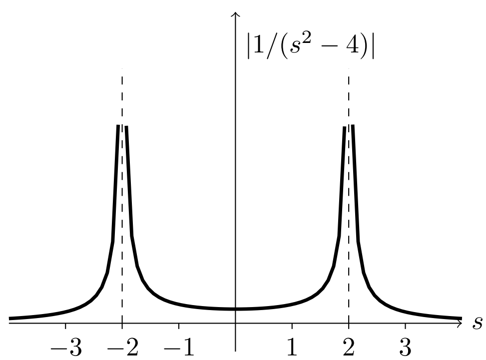

Figure 2: Graph of $\frac{1}{|s^2-4|}$ for s real.

As noted, the vertical asymptotes occur at values of s where the denominator of our function is 0. These are what we defined as the poles.

- $F_1(s) = \frac{1}{s}$ has a single pole at $s = 0$.
- $F_2(s) = \frac{1}{s^2-4}$ has two poles, one each at $s = \pm 2$.

Looking at Figures 1 and 2 you might be reminded of a tent. The poles of the tent are exactly the vertical asympotes which sit at the poles of the function.

Let's now try to graph $|F_1(s)|$ and $|F_2(s)|$ when we allow s to be complex. If $s = a + ib$ then $F_1(s)$ depends on two variables a and b, so the graph requires three dimensions: two for a and b, and one more (the vertical axis) for the value of $|F_1(s)|$. The graphs are shown in Figure 3 below. They are 3D versions of the graphs above in Figures 1 and 2. At each pole there is a conical shape rising to infinity, and far from the poles the function fall off to 0.

Figure 3: The graphs of $|1/s|$ and $1/|s^2 - 4|$.

Roughly speaking, the poles tell you the shape of the graph of a function $|F(s)|$: it is *large near the poles*. In the typical pole diagams seen in practice, the $|F(s)|$ is also small far away from the poles.

4. Poles and Exponential Growth Rate

If $a > 0$, the exponential function $f_1(t) = e^{at}$ grows rapidly to infinity as $t \to \infty$. Likewise the function $f_2(t) = e^{at} \sin bt$ is oscillatory with the amplitude of the oscillations growing exponentially to infinity as $t \to \infty$. In both cases we call a the *exponential growth rate* of the function.

The formal definition is the following

Definition: The **exponential growth rate** of a function $f(t)$ is the smallest value a such that

$$\lim_{t\to\infty} \frac{f(t)}{e^{bt}} = 0 \quad \text{for all } b > a. \tag{1}$$

In words, this says $f(t)$ grows slower than any exponential with growth rate larger than a.

Examples.

1. e^{2t} has exponential growth rate 2.

2. e^{-2t} has exponential growth rate -2. A negative growth rate means that the function is *decaying* exponentially to zero as $t \to \infty$.

3. $f(t) = 1$ has exponential growth rate 0.

4. $\cos t$ has exponential growth rate 0. This follows because $\lim_{t\to\infty} \frac{\cos t}{e^{bt}} = 0$ for all positive b.

5. $f(t) = t$ has exponential growth rate 0. This may be surprising because $f(t)$ grows to infinity. But it grows linearly, which is slower than any positive exponential growth rate.

6. $f(t) = e^{t^2}$ does not have an exponential growth rate since it grows faster than any exponential.

Poles and Exponential Growth Rate
We have the following theorem connecting poles and exponential growth rate.

Theorem: The exponential growth rate of the function $f(t)$ is the largest real part of all the poles of its Laplace transform $F(s)$.

Examples. We'll check the theorem in a few cases.

1. $f(t) = e^{3t}$ clearly has exponential growth rate equal to 3. Its Laplace transform is $1/(s-3)$ which has a single pole at $s = 3$,and this agrees with the exponential growth rate of $f(t)$.
2. Let $f(t) = t$, then $F(s) = 1/s^2$. $F(s)$ has one pole at $s = 0$. This matches the exponential growth rate zero found in (5) from the previous set of examples.
3. Consider the function $f(t) = 3e^{2t} + 5e^t + 7e^{-8t}$. The Laplace transform is $F(s) = 3/(s-2) + 5/(s-1) + 7/(s+8)$, which has poles at $s = 2$, 1, -8. The largest of these is 2. (Don't be fooled by the absolute value of -8, since $2 > -8$, the largest pole is 2.) Thus, the exponential growth rate is 2. We can also see this directly from the formula for the function. It is clear that the $3e^{2t}$ term determines the growth rate since it is the dominant term as $t \to \infty$.
4. Consider the function $f(t) = e^{-t}\cos 2t + 3e^{-2t}$ The Laplace transform is $F(s) = \frac{s}{(s+1)^2+4} + \frac{3}{s+2}$. This has poles $s = -1 \pm 2i$, -2. The largest real part among these is -1, so the exponential growth rate is -1.

Note that in item (4) in this set of examples the growth rate is negative because $f(t)$ actually *decays* to 0 as $t \to \infty$. We have the following
Rule:
1. If $f(t)$ has a negative exponential growth rate then $f(t) \to 0$ as $t \to \infty$.
2. If $f(t)$ has a positive exponential growth rate then $f(t) \to \infty$ as $t \to \infty$.

5. An Example of What the Poles Don't Tell Us

Consider an arbitrary function $f(t)$ with Laplace transform $F(s)$ and $a > 0$. Shift $f(t)$ to produce $g(t) = u(t-a)f(t-a)$, which has Laplace transform $G(s) = e^{-as}F(s)$. Since e^{-as} does not have any poles, $G(s)$ and $F(s)$ have exactly the same poles. That is, the poles can't detect this type of shift in time.