Part II Problems and Solutions

Problem 1: *[Poles]* **(a)** *For each of the pole diagrams below:*
(i) Describe common features of all functions $f(t)$ whose Laplace transforms have the given pole diagram.
(ii) Write down two examples of such $f(t)$ and $F(s)$.

The diagrams are: (1) $\{1, i, -i\}$. (2) $\{-1+4i, -1-4i\}$. (3) $\{-1\}$. (4) The empty diagram.

(b) *A mechanical system is discovered during an archaeological dig in Ethiopia. Rather than break it open, the investigators subjected it to a unit impulse. It was found that the motion of the system in response to the unit impulse is given by $w(t) = u(t)e^{-t/2}\sin(3t/2)$.*

(i) What is the characteristic polynomial of the system? What is the transfer function $W(s)$?

(ii) Sketch the pole diagram of the system.

(ii) The team wants to transport this artifact to a museum. They know that vibrations from the truck that moves it result in vibrations of the system. They hope to avoid circular frequencies to which the system response has the greatest amplitude. What frequency should they avoid?

(iv) Invoke the Mathlet `Amplitude Response and Pole Diagram`*, and set the system parameters* `b` *and* `k` *to the values of b/m and k/m you found in (i). Check to see that the amplitude response curve shows a maximum where you predicted it would. Grab the 3D image and move it to view the graph of $|W(s)|$ from different angles. Explain in words what each of the following graphical features in the 3D window represents: The yellow box-like figure; the green box-like figure; the yellow curve forming the base of the yellow boxlike figure; the red arrows; the yellow diamonds.*

Solution: **(a)** (1) $\{1, i, -i\}$: For large t, these functions have exponential growth rate e^t. (This means that for any $a < 1 < c$, $e^{at} < |f(t)| < e^{ct}$ for large t.) They also show a small oscillation of approximately constant amplitude and circular frequency 1. Examples: $f(t) = ae^t + b\sin(t)$ ($a, b \neq 0$) with $F(s) = \dfrac{a}{s-1} + \dfrac{b}{s^2+1}$.

(2) $\{-1+4i, -1-4i\}$: For large t, these functions show exponential decay like e^{-t}, and oscillate with circular frequency 4.
Examples: $f(t) = ae^{-t}\sin(4t)$ ($a \neq 0$) with $F(s) = \dfrac{4a}{(s+1)^2+16}$.

(3) $\{-1\}$: For large t, these functions decay like e^{-t} and do not oscillate. Examples: $f(t) = ae^{-t}$ ($a \neq 0$) with $F(s) = \dfrac{a}{s+1}$.

(4) No poles: For large t, these functions decay to zero faster than any exponential. Ex-

amples: $f(t) = a\delta(t-b)$ ($a \neq 0, b \geq 0$) with $F(s) = ae^{-bt}$, or $f(t) = a(u(t) - u(t-b))$ ($a \neq 0, b > 0$) with $F(s) = a\dfrac{1-e^{-bs}}{s}$.

(b) (i) Method I: For $t > 0$, $w(t)$ is a solution to the homogeneous equation. The roots must be $-\frac{1}{2} \pm \frac{3}{2}i$, so $p(s) = m\left(s - (-\frac{1}{2} + \frac{3}{2}i)\right)\left(s - (-\frac{1}{2} - \frac{3}{2}i)\right) = m(s^2 + s + \frac{5}{2})$. To find m, remember that $\dot{w}(0+) = \frac{1}{m}$ (for a second order system). $\dot{w}(t) = u(t)e^{-t/2}(\frac{3}{2}\cos(3t/2) - (-\frac{1}{2}\sin(3t/2))$, so $\dot{w}(0+) = \frac{3}{2}$, $m = \frac{2}{3}$, and $p(s) = \frac{2}{3}(s^2 + s + \frac{5}{2})$.

Method II: $W(s) = \mathcal{L}(w(t)) = \dfrac{3/2}{(s+(1/2))^2 + (9/4)} = \dfrac{1}{(2/3)(s^2+s+(5/2))}$, so $p(s) = \frac{2}{3}(s^2 + s + \frac{5}{2})$.

(ii) $\{-\frac{1}{2} \pm \frac{3}{2}i\}$.

(iii) This is a throwback problem. The complex gain is $W(i\omega) = \dfrac{3/2}{((5/2) - \omega^2) + i\omega}$, so the gain is $|W(i\omega)| = \dfrac{3/2}{\sqrt{((5/2)-\omega^2)^2 + \omega^2}}$. This is maximized when the denominator, or its square $(\frac{5}{2} - \omega^2)^2 + \omega^2$, is minimized. The derivative with respect to ω is $2(\frac{5}{2} - \omega^2)(-2\omega) + 2\omega$, which has roots at $\omega = 0$ and $\omega = \pm\sqrt{2}$. So $\omega_r = \sqrt{2}$ is the worst frequency.

(iv) The yellow box lies in the plane above the imaginary axis. The base is the amplitude response curve. The green box lies in the plane above the real axis. Its top lies on the graph of $|W(s)|$, and its base is the real axis. The red arrows lie above the poles of $W(s)$. The yellow diamonds are located at $(\pm i\omega, |W(i\omega)|)$, and represent the chosen value of the input circular frequency and the corresponding gain.