I just recalling some of the notation we are going to need for today, and a couple of the facts that we're going to use, plus trying to clear up a couple of confusions that the recitations report. This can be thought of two ways. It's a formal polynomial in D, in the letter D. It just has the shape of the polynomial, D squared plus AD plus B.

A and B are constant coefficients.

But, it's also, at the same time, if you think what it does, it's a linear operator on functions. It's a linear operator on functions like y of t. You think of it both ways: formal polynomial because we want to do things like factoring it, substituting two for D and things like that.

Those are things you do with polynomials.

You do them algebraically. You can take the formal derivative of the polynomial because it's just sums of powers. On the other hand, as a linear operator, it does something to functions.

It differentiates them, multiplies them by constants or something like that. So it's, so to speak, has a dual aspect this way. And, that's one of the things we are exploiting what we use operator methods to solve differential equations. Now, let me remind you of the key thing we were interested in. f of t: not any old function, we'll get to that next time, but f of t, exponentials.

So, it should be an exponential or something like an exponential, or pretty close to it, for example, something with sine t and cosine t, or e to the, that could be thought of as part of the real or imaginary part of a complex exponential.

And, maybe by the end of today, we will have generalized that even little more. But basically, I'm interested in exponentials. Let's make it alpha complex.

That will at least take care of the cases, e to the ax times cosine bx, sin bx, which are the main cases. Those are the main cases.

Then, remember the little table we made.

I simply gave you the formula for the particular solution.

So, what we're looking for is we already know how to solve the homogeneous equation. What we want is that particular solution. And then, the recipe for it I gave you, these things were proved by the substitution rules and exponential shift rules. The recipe was that if f of t was, let's make a little table.

f of t is, well, it's always e to the a t.

So, in other words, it's e to the a t. The cases are, so yp, what is the yp? Well, it is the normal case is yp equals e to that alpha t divided by the polynomial where you substitute, you take that polynomial, and wherever you see a D, you substitute the complex number, alpha. There, I'm thinking of it as a formal polynomial. I'm not thinking of it as an operator. Now, this breaks down.

So, that's the formula for the particular solution.

The only trouble is, it breaks down if p of alpha is zero. So, we have to assume that it's not. Now, if p of alpha is zero, that means alpha is a root of the polynomial, a zero of the polynomial is a better word.

So, in that case, it will be e to the alpha t divided by p prime of alpha.

Differentiate formally the polynomials, -- -- and you will get 2D plus A.

And now, substitute in the alpha.

And, this will be okay provided p prime of alpha is not zero. That means that alpha is the simple root, simple zero of p. And then, there's one more case, which, since I won't need today, I won't write on the board. But, you'll need it for homework. So, make sure you know it.

Another words, if this is zero, then you've got a double root. And, there is still a different formula. And, this is wrong because I forgot the t. Yes?

I could tell on your faces. That was before, and now we are up to today. What we are interested in talking about today is what this has to do with the phenomenon of resonance. Everybody knows at least one case of resonance, I hope.

A little kid is on his swing, right?

Back and forth, and they are very, very little, so they want a push.

Okay, well, everybody knows that to make the swing go, a swing has a certain natural frequency.

It swings back and forth like that.

It's a simple pendulum. It's actually damped, but let's pretend that it isn't.

Everybody knows you want to push a kid on a swing so that they go high. You have to push with essentially the same frequency that the natural frequency of the spring, of the swing is. It's automatic, because when you come back here, it gets to there, and that's where you push. So, automatically, you time your pushes. But if you want the kid to stop, you just do the opposite. Push at the wrong time.

So anyway, that's resonance. Of course, there are more serious applications of it. It's what made the Tacoma Bridge fall down, and I think movies of that are now being shown not merely on television, but in elementary school. Resonance is what made, okay, more resonance stories later.

So, my aim is, what is this physical phenomenon, that to get a big amplitude you should have it match the frequency? What does that have to do with a differential equation? Well, the differential equation for that simple pendulum, let's assume it's undamped, will be of the type y double prime plus, I'm using t now since t is time.

That will be our new independent variable, plus omega nought squared is the natural frequency of the pendulum or of the spring, or whatever it is that's doing the vibrating. Yeah, any questions?

What we're doing is driving that with the cosine, with something of a different frequency.

So, this is the input, or the driving term as it's often called, or it's sometimes called the forcing term. And, the point is I'm going to assume that the frequency is different.

The driving frequency is different from the natural frequency. So, this is the input frequency. Okay, and now let's simply solve the equation and see what we get.

So, it's if I write it using the operator, it's D squared plus omega nought squared applied to y is equal to cosine.

It's a good idea to do this because the formulas are going to ask you to substitute into a polynomial.

So, it's good to have the polynomial right in front of you to avoid the possibility of error.

Well, really what I want is the particular solution.

It's the particular solution that's going to give me a pure oscillation. And, the thing to do is, of course, since this cosine, you want to make it complex.

So, we are going to complexify the equation in order to be able to solve it more easily, and in order to be able to use those formulas. So, the complex equation is going to be D squared plus omega nought squared.

Well, it's going to be a complex, particular solution. So, I'll call it y tilde.

And, on the right-hand side, that's going to be e to the i omega1 t. Cosine is the real part of this. So, when we get our answer, we want to be sure to take the real part of the answer.

I don't want the complex answer, I want its real part.

I want the real answer, in other words, the really real answer, the real real answer.

So, now without further ado, because of those beautiful, the problem has been solved once and for all by using the substitution rule. I did that for you on Monday.

The answer is simply e to the i omega1 t divided by what? This polynomial with omega one substituted in for D. So, sorry, i omega one, the complex coefficient of t.

So, it is substitute i omega for D, I omega one for D, and you get (i omega one) squared plus omega nought squared.

Well, let's make that look a little bit better.

This should be e to the (i omega one t) divided by, now, what's this?

This is simply omega nought squared minus omega one squared. But, I want the real part of it. So, as one final, last step, the real part of that is what we call just the real particular solution, so, yp without the tilde anymore. And, the real part of this, well, this cosine plus i sine. And, the denominator, luckily, turns out to be real. So, it's simply going to be cosine omega one t.

That's the top, divided by this thing, omega nought squared minus omega one squared.

In other words, that's the response. This is the input, and that's what came out. Well, in other words, what one sees is, regardless of what natural frequency this system wanted to use for itself, at least for this solution, what it responds to is the driving frequency, the input frequency.

The only thing is that the amplitude has changed, and in a rather dramatic way, if omega1, depending on the relative sizes of omega1 and omega2.

Now, the interesting case is when omega one is very close to omega, the natural frequency. When you push it with approximately it's natural frequency, then the solution is big amplitude. The amplitude is large.

So, the solution looks like the frequency.

The input might have looked like this.

Well, it's cosine, so it ought to start up here.

The input might have looked like this, but the response will be a curve with the same frequency and still a pure oscillation. But, it will have much, much bigger amplitude. And, it's because the denominator, omega nought squared minus omega one squared, is always zero.

So, the response will, instead, look like this.

Now, to all intents and purposes, that's resonance.

You are pushing something with approximately the same frequency, something that wants to oscillate.

And, you are pushing it with approximately the same frequency that it would like to oscillate by itself.

And, what that does is it builds up the amplitude Well, what happens if omega one is actually equal to omega zero? So, that's the case I'd like to analyze for you now. Suppose the two are equal, in other words. Well, the problem is, of course, I can't use that same solution.

It isn't applicable. But that's why I gave you, derived for you using the exponential shift law last time, the second version, when it is a root.

So, if omega one equals omega nought, so now our equation looks like D squared plus omega nought squared, the natural frequency, y.

But this time, the driving frequency, the input frequency, is omega nought itself.

Then, the same analysis, a lot of it is, well, I'd better be careful. I'd better be careful.

Let's go through the analysis again very rapidly.

What we want to do is first complexify it, and then solve. So, the complex equation will be D squared plus omega nought squared times y tilde equals e to the i omega nought t, this time.

But now, i omega is zero of this polynomial.

That's why I picked it, right?

If I plug in i omega zero, I get i omega zero quantity squared plus omega nought squared.

That's zero.

So, I'm in the second case. So, i omega nought is a simple root, simple zero, of D squared plus omega nought, that polynomial squared.

Therefore, the complex particular solution is now t e to the i omega nought t divided by p prime, where you plug in that root, the i omega nought.

Now, what's p prime?

p prime is 2D, right?

If I differentiate this formally, as if D were a variable, the way you differentiate polynomials, the derivative, this is a constant, and the derivative is 2D. So, the denominator should have two times for D. You are going to plug in i omega zero. So, it's 2 i omega zero.

And now, I want the real part of that, which is what? Well, think about it.

The top is cosine plus i sine. The real part is now going to come from the sine, right, because it's cosine plus i sine. But this i is going to divide out the i that goes with this sine.

And, therefore, the real part is going to be t times the sine, this time, of omega nought t.

And, that's going to be divided by, well, the i canceled out the i that was in front of the sine function. And therefore, what's left is two omega nought down below.

So, that's our particular solution now.

Well, it looks different from that guy.

It doesn't look like that anymore.

What does it look like? Well, it shows the way to plot such things is basically it's an oscillation of frequency omega nought. But, its amplitude is changing.

So, the way to do it is, as always, if you have a basic oscillation which is neither too fast nor too slow, think of that as the thing, and the other stuff multiplying it, think of it as changing the amplitude of that oscillation with time. So, the amplitude is that function, t divided by two omega zero.

So, just as we did when we talked about damping, you plot that and it's negative on the picture.

So, this is the function whose graph is t divided by two omega nought. That's the changing amplitude, as it were. And then, the function itself does what oscillation it can, but it has to stay within those lines. So, the thing that's oscillating is sine omega nought t, which would like to be a pure oscillation, but can't because its amplitude is being changed by that thing.

So, it's doing this, and now the rest I have to leave to your imagination. In other words, what happens when omega nought is equal to, when the driving frequency is actually equal to omega nought, mathematically this turns into a different looking solution, one with steadily increasing amplitude.

The amplitude increases linearly like the function t divided by two omega nought.

Well, many people are upset by this, slightly, in the sense that there is a funny feeling.

How is it that that solution can turn into this one?

If I simply let omega one go to omega zero, what happens?

Well, the pink curve just gets taller and taller, and after a while all you see of it is just a bunch of vertical lines which seem to be spaced at whatever the right period is for that function. It's sort of like being in a first story window and watching a giraffe go by.

All you see is that. Okay.

So, my concern is how does that function turn into this one?

I have something in mind to remind you of, and that's why we'll go through this little exercise.

It's a simple exercise. But the function of it is, of course that as omega one goes to omega zero cannot possibly turn into this. It's doing the wrong thing near zero. It's already zooming up.

But, the point is, this is not the only particular solution on the block. Any solution whatsoever of the differential equation, the inhomogeneous equation, is a particular solution. It's like Fred Rogers: everybody is special. Okay, so all solutions are special. We don't have to use that one.

So, I will use, where are all the other solutions? So, I'm going back to the equation D squared plus omega zero squared, applied to y, is equal to cosine omega one t.

Now, the particular solution we found was that one, cosine omega one t divided by that omega nought squared minus omega one squared.

What do the other particular solutions look like?

Well, in general, any particular solution will look like that one we found, what is it, omega nought squared minus omega one squared, plus I'm allowed to add to it any piece of the complementary solution. Equally particular, and equally good, as a particular solution is this plus anything which solved the homogeneous equation.

Now, all I'm going to do is pick out one good function which solves the homogeneous equation, and here it is.

It's the function minus cosine. In fact, what does solve the homogeneous equation? Well, it's solved by sine omega nought t, cosine omega nought t, and any linear combination of those. So, out of all those functions, the one I'm going to pick is cosine omega nought t.

And, I'm going to divide it by this same guy.

So, this is part of the complementary solution.

That's what we call the complementary solution, the solution to the associated homogeneous equation, to the reduced equation. Call it what you like.

So, this is one of the guys in there, and it's still a particular solution to take the one I first found, and add to it anything which solves the homogeneous equation.

I showed you that when we first set out to solve the inhomogeneous equation in general.

Now, why do I pick that? Well, I'm going to now calculate, what's the limit? So, these guys are also good solutions to that. This is a good solution to that equation, this equation. All I'm going to do now is calculate the limit as omega one approaches omega zero of this function. Well, what is that?

It's cosine omega one t minus cosine omega zero t divided by omega nought squared minus omega one squared.

Now, you see why I did that. If I let just this guy, omega one approaches omega zero, I get infinity. I don't get anything.

But, this is different here because I fixed it up, now. The denominator becomes zero, but so does the numerator. In other words, I've put myself in position to use L'Hopital rule.

So, let's L'Hopital it. It's the limit.

As omega one approaches omega zero, and what do you do?

You differentiate the top and the bottom with respect to what?

Right, with respect to omega one.

Omega one is the variable. That's what's changing.

The t that I'm thinking of is, I'm thinking, for the temporary fixed. This has a fixed value.

Omega nought is fixed. All that's changing in this limit operation is omega one. And therefore, it's with respect to omega one that I differentiate it.

You got that? Well, you are in no position to say yes or no, so I shouldn't even ask the question, but okay, rhetorical question.

All right, let's differentiate this expression, the top and bottom with respect to omega one.

So, the derivative of the top with respect to omega one is negative sine omega one t.

But, I have to use the chain rule.

That's differentiating with respect to this argument, this variable. But now, I must take times the derivative of this thing with respect to omega one.

And that is t is the constant, so times t.

And, how about the bottom? The derivative of the bottom with respect to omega one is, well, that's a constant.

So, it becomes zero. And, this becomes negative two omega one. So, it's the limit of this expression as omega one approaches omega zero.

And now it's not indeterminate anymore. The answer is, the negative signs cancel. It's simply t sine omega nought t divided by two omega nought.

So, that's how we get that solution. It is a limit as omega one, but not of the particular solution we found first, but of this other one. Now, it's still too much algebra. I mean, what's going on here?

Well, that's something else you should know.

Okay, so my question is, therefore, what does this mean?

What's the geometric meaning of all this?

In other words, what does that function look like? Well, that's another trigonometric identity, which in your book is just buried as half of one line sort of casual as if everybody knows it, and I know that virtually no one knows it.

But, here's your chance. So, the cosine of B minus the cosine of A can be expressed as a product of signs. It's the sine of (A minus B) over two times the sine of (A plus B) over two, I believe.

My only uncertainty: is there a two in front of that? I think there has to be.

Let me check. Sorry.

Is there a two? I wouldn't trust my memory anyway. I'd look it up.

I did look it up, two, yes.

If you had to prove that, you could use the sine formula to expand this out. That would be a bad way to do it. The best way is to use complex numbers. Express the sign in terms of complex numbers, exponentials, you know, the backwards Euler formula.

Then do it here, and then just multiply those two expressions involving exponentials together, and cancel, cancel, cancel, cancel, cancel, and this is what you will end up with.

You see why I did this. It's because this has that form. So, let's apply that formula to it. So, what's the left-hand side?

B is omega one t, and A is omega nought t.

So, this is omega one t, and this is omega nought t All right, so what we get is that the cosine of omega one t minus the cosine of omega nought t, which is exactly the numerator of this function that I'm trying to get a handle on.

Then we will divide it by its amplitude.

So, that's this constant factor that's real.

It's a small number because I'm thinking of omega one as being rather close to omega zero, and getting closer and closer. What does this tell us about the right-hand side? Well, the right-hand side is twice the sine of A minus B.

Now, that's good because these guys sort of resemble each other. So, that's (omega nought minus omega one) times t.

That's A minus B, and I'm supposed to divide that by two. And then, the other one will be the same thing with plus: sine omega nought plus omega one over two times t.

Now, how big is this, approximately?

Remember, think of omega one as close to omega zero.

Then, this is approximately omega zero. So this part is approximately sine of omega zero t.

This part, on the other hand, that's a very small thing.

Okay, now what I want to know is what does this function look like? The interest in knowing what the function looks like it is because we want to be able to see that it's limited is that thing.

You can't tell what's what its limit is, geometrically, unless you know it looks like. So, what does it look like?

Well, again, the way to analyze it is the thing, that thing. What you think of is, yeah, of course you cannot divide one side of equality without dividing the equation by the other side.

So, that's got to be there, too.

Now, what does that look like? Well, the way to think of it is, here is something with a normal sort of frequency, omega nought. It's doing its thing.

It's a sine curve. It's doing that.

What's this? Think of all this part as varying amplitude. It's just another example of what I gave you before. Here is a basic, pure oscillation, and now, think of everything else that's multiplying it as varying its amplitude.

All right, so what does that thing look like?

Well, first what we want to do is plot the amplitude lines.

Now, what will they be? This is sine of an extremely small number times t. The frequency is small.

How does the sine curve look if its frequency is very low, very close to zero? Well, that must mean its period is very large. Here's something with a big frequency. Here's something with a very, very low frequency. Now, with a low frequency, it would hardly get off the ground and get up to one here, and it would do that. But, it's made to look a little more presentable because of this coefficient in front, which is rather large. And so, what this thing looks like, I won't pause to analyze it more exactly.

It's something which goes up at a reasonable rate for quite a while, and let's say that's quite awhile.

And then it comes down, and then it goes, and so on. Of course, in figuring out its amplitude, we have to be willing to draw its negative, too. And since I didn't figure things out right, I can at least make it cross, right? Okay.

So, this is a picture of this slowly varying amplitude.

And in between, this is the function which is doing the oscillation, as well as it can.

But, it has to stay within that amplitude.

So, it's doing this. Now, what happens?

As omega one approaches omega zero, this frequency gets closer and closer to zero, which means the period of that dotted line gets further and further out, goes to infinity, and you never do ultimately get a chance to come down again. All you can see is the initial part, where it's rising and rising.

And, that's how this curve turns into that one.

Now, of course, this curve is enormously interesting. You must have had this somewhere. That's the phenomenon of what are called beats. Too frequencies-- Your book has half a page explaining this.

That's the half a page where he gives you this identity, except it gives it in a wrong form, so that it's hard to figure out. But anyway, the beats are two frequencies when you combine them, the two frequencies being two combined pure oscillations where the frequencies are very close to each other. What you get is a curve which looks like that. And, of course, what you hear is the envelope of the curve.

You hear the dotted lines. Well, you hear this.

You hear that, too.

But, what you hear is-- And, that's how good violinists and cellists, and so on, tune their instruments.

They get one string right, and then the other strings are tuned by listening. They don't actually listen for the sound of the note. They listened just for the beats, wah, wah, wah, wah, and they turn the peg and it goes wah, wah, wah, wah, and then finally as soon as the wahs disappear, they know that the two strings are in tune.

A piano tuner does the same thing.

Of course, I, being a very bad cellist, use a tuner. That's another solution, a more modern solution. Okay.

Oh well. Let's give it a try.

The bad news is that problem six in your problem set, I didn't ask you about the undamped case.

I thought, since you are mature citizens, you could be asked about the damped case.

I warn you, first of all you have to get the notation.

This is probably the most important thing I'll do with this. Your book uses this, resonance.

I'm optimistic. [LAUGHTER] Let's say zero or f of t. It doesn't matter.

In other words, the constants, the book uses two sets of constants to describe these equations. If it's a spring, and not even talking about RLC circuits, the spring mass, damping, k, spring constant. Then you divide out by m and you get this. You're familiar with that.

And, it's only after you divided out by the m that you're allowed to call this the square of the natural frequency.

So, omega naught is the natural frequency, the natural undamped frequency. If this term were not there, that omega nought would give the frequency with which the system, the little spring would like to vibrate by itself. Now, further complication is that the visual uses neither of these.

The visual uses x double dot plus b times x prime, I think we will have to fix this in the future, but for now, just live with it, plus kx, and that's some function, again, a function. So, in other words, the problem is that b is okay, can't be confused with c.

On the other hand, this is not the same k as that.

What I'm trying to say is, don't automatically go to a formula one place, and assume it's the same formula in another place. You have to use these equivalences. You have to look and see how the basic equation was written, and then figure out what the constant should be. Now, there was something called, when we analyzed this before, and this has happened in recitation, there was the natural, damped frequency.

I'll call it the natural, damped frequency.

The book calls it the pseudo-frequency.

It's called pseudo-frequency because the function, if you have zero on the right hand side, but have damping, the function isn't periodic. It decays.

It does this. Nonetheless, it still crosses the t-axis at regular intervals, and therefore, almost everybody just casually refers to it as the frequency, and understands it's the natural damped frequency. Now, the relation between them is given by the little picture I drew you once.

But, I didn't emphasize it enough.

Here is omega nought.

Here is the right angle. The side is omega one, and this side is the damping.

So, in other words, this is fixed because it's fixed by the spring. That's the natural frequency of the spring, by itself. If you are damping near the motion, then the more you damped it, the bigger this side gets, and therefore the smaller omega one is, the bigger the damping, then the smaller the frequency with which the damped thing vibrates. That sort of intuitive, and vice versa. If you decrease the damping to almost zero, well, then you'll make omega one almost the same size as omega zero.

This must be a right angle, and therefore, if there's very little damping, the natural damped frequency will be almost the same as the original frequency, the natural frequency. So, the relation between them is that omega one squared is equal to omega nought squared minus p squared, and this comes from the characteristic roots from the characteristic roots of the damped equation.

So, we did that before. I'm just reminding you of it.

Now, the third frequency which now enters, and that I'm asking you about on the problem set is if you've got a damped spring, okay, what happens when you impose a motion on it with yet a third frequency? In other words, drive the damped spring. I don't care.

I switched to y, since I'm in y mode.

So, our equation looks like this, just as it did before, except now going to drive that with an undetermined frequency, cosine omega t.

And, my question, now, is, see, it's not going to be able to resonate in the correct-- you really only get true resonance when you don't have damping.

That's the only time where the amplitude can build up indefinitely. But nonetheless, for all practical purposes, and there's always some damping unless you are a perfect vacuum or something, there's almost always some damping.

So, p isn't zero, can't be exactly zero.

So, the problem is, which omega gives, which frequency in the input, which input frequency gives the maximal amplitude for the response?

We solved that problem when it was undamped, and the answer was easy. Omega should equal omega zero.

But, when it's damped, the answer is different.

And, I'm not asking you to do it in general.

I'm giving you some numbers. But nonetheless, it still must be the case. So, I'm giving you, I give you specific values of p and omega zero.

That's on the problem set. Of course, one of them is tied to your recitation. But, the answer is, I'm going to give you the general formula for the answer to make sure that you don't get wildly astray.

Let's call that omega r, the resonant omega. This isn't true resonance.

Your book calls it practical resonance.

Again, most people just call it resonance.

So, you know what I mean, type of thing.

It is omega r is very much like that.

Maybe I should have written this one down in the same form.

Omega one is the square root of omega nought squared minus p squared.

What would you expect? Well, what I would expect is that omega r should be omega one.

The damped system has a natural frequency.

The resonant frequency should be the same as that natural frequency with which the damped system wants to do its thing.

And the answer is, that's not right.

It is the square root. It's a little lower.

It's a little lower. It is omega nought squared minus two p squared.