This is a brief, so, the equation, and we got the characteristic equation from the last time.

The general topic for today is going to be oscillations, which are extremely important in the applications and in everyday life. But, the oscillations, we know, are associated with a complex root.

So, they correspond to complex roots of the characteristic equation. r squared plus br plus k equals zero. I'd like to begin.

Most of the lecture will be about discussing the relations between these numbers, these constants, and the various properties that the solutions, oscillatory solutions, have.

But, before that, I'd like to begin by clearing up a couple of questions almost everybody has at some point or other when they study the case of complex roots.

Complex roots are the case which produce oscillations in the solutions. That's the relation, and that's why I'm talking about this for the first few minutes. Now, what is the problem?

The complex roots, of course, there will be two roots, and they occur at the complex conjugates of each other. So, they will be of the form a plus or minus bi. Last time, I showed you, I took the root r equals a plus bi, which leads to the solution. The corresponding solution is a complex solution which is e to the at, (a plus i b)t.

And, what we did was the problem was to get real solutions out of that.

We needed two real solutions, and the way I got them was by separating this into its real part and its imaginary part.

And, I proved a little theorem for you that said both of those give solutions. So, the real part was e to the a t times cosine b t, and the imaginary part was e to the at sine b t.

And, those were the two solutions.

So, here was y1. And, the point was those, out of the complex solutions, we got real solutions.

We have to have real solutions because we live in the real world. The equation is real.

Its coefficients are real. They represent real quantities.

That's the way the solutions, therefore, have to be.

So, these, the point is, these are now real solutions, these two guys, y1 and y2.

Now, the first question almost everybody has, and I was pleased to see at the end of the lecture, a few people came up and asked me, yeah, well, you took a plus bi, but there was another root, a minus bi. You didn't use that one.

That would give two more solutions, right?

Of course, they didn't say that.

They were too smart. They just said, what about that other root? Well, what about it?

The reason I don't have to talk about the other root is because although it does give to solutions, it doesn't give two new ones. Maybe I can indicate that most clearly here even though you won't be able to take notes by just using colored chalk. Suppose, instead of plus bi, I used a minus bi.

What would have changed? Well, this would now become minus here. Would this change?

No, because e to the minus ibt is the cosine of minus b, but that's the same as the cosine of b.

How about here? This would become the sine of minus bt. But that's simply the negative of the sine of bt. So, the only change would have been to put a minus sign there. Now, I don't care if I get y2 or negative y2 because what am I going to do with it?

When I get it, I'm going to write y, the general solution, as c1 y1 plus c2 y2.

So, if I get negative y2, that just changes that arbitrary constant from c2 to minus c2, which is just as arbitrary a constant. So, in other words, there's no reason to use the other root because it doesn't give anything new. Now, there the story could stop. And, I would like it to stop, frankly, but I don't dare because there's a second question. And, I'm visiting recitations not this semester, but in previous semesters.

In 18.03, so many recitations do this.

I have to partly inoculate you against it, and partly tell you that some of the engineering courses do do it, and therefore you probably should learn it also.

So, there is another way of proceeding, which is what you might have thought. Hey, look, we got two complex roots. That gives us two solutions, which are different. Neither one is a constant multiple of the other. So, the other approach is, use, as a general solution, y equals, now, I'm going to put a capital C here.

You will see why in just a second, times e to the (a plus b i) times t.

And then, I will use the other solution: C2 times e to the (a minus b i) t.

These are two independent solutions.

And therefore, can't I get the general solution in that form? Now, in a sense, you can. The whole problem is the following, of course, that I'm only interested in real solutions. This is a complex function.

This is another complex function.

It's got an i in it, in other words, when I write it out as u plus iv.

If I expect to be able to get a real solution out of that, that means I have to make, allow these coefficients to be complex numbers, and not real numbers.

So, in other words, what I'm saying is that an expression like this, where the a plus bi and a minus bi are complex roots of that characteristic equation, is formally a very general, complex solution to the equation. And therefore, the problem becomes, how, from this expression, do I get the real solutions? So, the problem is, I accept these as the complex solutions.

My problem is, to find among all these guys where C1 and C2 are allowed to be complex, the problem is, which of the green solutions are real?

Now, there are many ways of getting the answer.

There is a super hack way. The super hack way is to say, well, this one is C1 plus i d1.

This is C2 plus i d2.

And, I'll write all this out in terms of what it is, you know, cosine plus i sine, and don't forget the e to the at. And, I will write it all out, and it will take an entire board.

And then, I will just see what the condition is.

I'll write its real part, and its imaginary part.

And then, I will say the imaginary part has got to be zero. And, then I will see what it's like. That works fine.

It just takes too much space. And also, it doesn't teach you a few things that I think you should know.

So, I'm going to give another. So, let's say we can answer this two ways: by hack, in other words, multiply everything out. Multiply all out, make the imaginary part equal zero.

Now, here's a better way, in my opinion.

What I'm trying to do is, this is some complex function, u plus iv. How do I know when a complex function is real? I want this to be real.

Well, the hack method corresponds to, say, v must be equal to zero. It's real if v is zero.

So, expand it out, and see why v is zero.

There's a slightly more subtle method, which is to change i to minus i. And, what?

And, see if it stays the same because if I change i to minus i and it turns out, the expression doesn't change, then it must have been real, if the expression doesn't change when I change I to minus I.

Well, sure. But you will see it works.

Now, that's what I'm going to apply to this.

If I want this to be real, I phrase the question, I rephrase the question for the green solution as change, so I'm going to change i to minus i in the green thing, and that's going to give me what conditions, and that will give conditions on the C's.

Well, let's do it. In fact, it's easier done than talked about. Let's change, take the green solution, and change.

Well, I better recopy it, C1.

So, these are complex numbers. That's why I wrote them as capital letters because little letters you tend to interpret as real numbers. So, C1 e to the (a plus b i)t, I'll recopy it quickly, plus C2 e to the (a minus b i).

Okay, we're going to change i to negative i.

Now, here's a complex number. What happens to it when you change i to negative i? You change it into its-- Class?

What do we change it to? Its complex conjugate.

And, the notation for complex conjugate is you put a bar over it. So, in other words, when I do that, the C1 changes to C1 bar, complex conjugate, the complex conjugate of C1.

What happens to this guy? This guy changes to e to the (a minus b i) t. This changes to the complex conjugate of

C2 now, times e to the (a plus b i) t.

Well, I want these two to be the same. I want the two expressions the same. Why do I want them the same?

Because, if there's no change, that will mean that it's real.

Now, when is that going to happen?

That happens if, well, here is this, that. If C2 should be equal to C1 bar, that's only one condition. There's another condition.

C2 bar should equal C1. So, I get two conditions, but there's really only one condition there because if this is true, that's true too. I simply put bars over both things, and two bars cancel each other out.

If you take the complex conjugate and do it again, you get back where you started. Change i to minus i, and then i to minus i again. Well, never mind.

Anyway, these are the same. This equation doesn't say anything that the first one didn't say already.

So, this one is redundant. And, our conclusion is that the real solutions to the equation are, in their entirety, I now don't need both C2 and C1.

One of them will do, and since I'm going to write it out as a complex number, I will write it out in terms of its coefficient. So, it's C1.

Let's just simply write it. C plus i times d, that's the coefficient. That's what I called C1 before.

And, that's times e to the (a plus b i) t.

There's no reason why I put bi here and id there, in case you're wondering, sheer caprice.

And what's the other term? Now, the other term is completely determined. Its coefficient must be C minus i d times e to the (a minus b i) t.

In other words, this thing is perfectly general. Any complex number times that first root you use, exponentiated, and the second term can be described as the complex conjugate of the first. The coefficient is the complex conjugate, and this part is the complex conjugate of that.

Now, it's in this form, many engineers write the solution this way, and physicists, too, so, scientists and engineers we will include.

Write the solution this way. Write the real solutions this way in that complex form. Well, why do they do something so perverse? You will have to ask them.

But, in fact, when we studied Fourier series, we will probably have to do something, have to do that at one point. If you work a lot with complex numbers, it turns out to be, in some ways, a more convenient representation than the one I've given you in terms of sines and cosines.

Well, from this, how would I get, suppose I insisted, well, if someone gave it to me in that form, I don't see how I would convert it back into sines and cosines. And, I'd like to show you how to do that efficiently, too, because, again, it's one of the fundamental techniques that I think you should know. And, I didn't get a chance to say it when we studied complex numbers that first lecture.

It's in the notes, but it doesn't prove anything since I don't think it made you use it in an example.

So, the problem is, now, by way of finishing this up, too, to change this to the old form, I mean the form involving sines and cosines. Now, again, there are two ways of doing it. The hack way is you write it all out. Well, e to the (a plus b i)t turns into e to the a t times cosine this plus i sine that. And, the other term does, too. And then you've got stuff out front. And, the thing stretches over two boards. But you group all the terms together. You finally get it.

By the way, when you do it, you'll find that the imaginary part disappears completely. It has to because that's the way we chose the coefficients. So, here's the hack method.

Write it all out: blah, blah, blah, blah, blah, blah, blah, and nicer.

Nicer, and teach you something you're supposed to know.

Write it this way. First of all, you notice that both terms have an e to the a t factor. Let's get rid of that right away. I'm pulling it out front because that's automatically real, and therefore, isn't going to affect the rest of the answer at all.

So, let's pull out that, and what's left?

Well, what's left, you see, involves just the two parameters, C and d, so I'm going to have a C term.

And, I'm going to have a d term.

What multiplies the arbitrary constant, C?

Answer: after I remove the e to the a t, what multiplies it is, e to the b i t plus e to the -- e to the b i t. Let's write it i b t.

And, the other term is plus e to the negative i b t.

See how I got that, pulled it out? And, how about the d?

What goes with d? d goes with, well, first of all, there's an I in front that i better not forget. And then, the rest of it is i.

So, it's i d times, it's e to the b i t, e to the i b t minus, now, e to the minus i b t.

So, that's the way the solution looks. It doesn't look a lot better, but now you must use the magic formulas, which, I want you to know as well as you know Euler's formula, even better than you know Euler's formula.

They're a consequence of Euler's formula.

They're Euler's formula read backwards.

Euler's formula says you've got a complex exponential here.

Here's how to write it in terms of sines and cosines.

The backwards thing says you've got a sine or a cosine.

Here is the way to write it in terms of complex exponentials.

And, remember, the way to do it is, cosine a is equal to e to the i a t, i a, plus e to the negative i a divided by two.

And, sine of a is almost the same thing, except you use a minus sign. And, what everybody forgets, you have to divide by i. So, this is a backwards version of Euler's formula. The two of them taken together are equivalent to Euler's formula.

If I took cosine a, multiply this through by i, and added them up, on the right-hand side I'd get exactly e to the ia. I'd get Euler's formula, in other words. All right, so, what does this come out to be, finally?

This particular sum of exponentials, you should always recognize as real.

You know it's real because when you change i to minus i, the two terms switch. And therefore, the expression doesn't change. What is it?

This part is twice the cosine of bt.

What's this part? This part is 2 i times the sine of bt. And so, what does the whole thing come to be? It is e to the a t times 2C cosine bt plus i times, did I lose possibly a, no it's okay, minus i times i is minus, so, minus 2d times the sine of bt.

Shall I write that out?

So, in other words, it's e to the a t times 2C cosine b t minus 2d times the sine of b t, which is, since 2C and negative 2d are just arbitrary constants, just as arbitrary as the constants of C and d themselves are.

This is our old form of writing the real solution.

Here's the way using science and cosines, and there's the way that uses complex numbers and complex functions throughout.

Notice they both have two arbitrary constants in them, C and d, two arbitrary constants.

That, you expect. But that has two arbitrary constants in it, too, just the real and imaginary parts of that complex coefficient, C plus i d.

Well, that took half the period, and it was a long, I don't consider it a digression because learning those ways of dealing with complex numbers of complex functions is a fairly important goal in this course, actually.

But let's get back now to studying what the oscillations actually look like.

Okay, well, I'd like to save a little time, but very quickly, you don't have to reproduce this sketch.

I remember very well from Friday to Monday, but I can't expect you to for a variety of reasons.

I mean, I have to think about this stuff all weekend.

And you, God forbid. So, here's the picture, and I won't explain anymore what's in it, except there's the mass. Here is the spring constant, the spring with its constant here.

Here's the dashpot with its constant.

The equation is from Newton's law: m x double, so this will be x, and here's, let's say, the equilibrium point is over here.

It looks like m x double prime; we derived this last time, plus c x prime plus k x equals zero.

And now, if I put that in standard form, it's going to look like x double prime plus c over m x prime plus k over m times x equals zero.

And, finally, the standard form in which your book writes it, which is good, it's a standard form in general that is used in the science and engineering courses.

One writes this as, just to be perverse, I'm going to change x back to y, okay, mostly just to be eclectic, to get you used to every conceivable notation.

So, I'm going to write this to change x to y.

So, that's going to become y double prime.

And now, this is given a new name, p, except to get rid of lots of twos, which would really screw up the formulas, make it 2p. You will see why in a minute.

So, there's 2p times y prime, and this thing we are going to call omega nought squared.

Now, that's okay. It's a positive number.

Any positive number is the square of some other positive number. Take a square root.

You will see why, it makes the formulas much pretty to call it that. And, it makes also a lot of things much easier to remember. So, all I'm doing is changing the names of the constants that way in order to get better formulas, easier to remember formulas at the end.

Now, we are interested in the case where there is oscillations. In other words, I only care about the case in which this has complex roots, because if it has just real roots, that's the over-damped case. I don't get any oscillations.

By far, oscillations are by far the more important of the cases, I mean, just because, I don't know, I could go on for five minutes listing things that oscillate, oscillations, you know, like this.

So they can oscillate by going to sleep, and waking up, and going to sleep, and waking up.

They could oscillate. So, that means we're going to get complex roots. The characteristic equation is going to be r squared plus 2p. So, p is a constant, now, right? Often, p I use in this position to indicate a function of t. But here, p is a constant.

So, r squared plus 2p times r plus omega nought squared is equal to zero.

Now, what are its roots? Well, you see right away the first advantage in putting in the two there.

When I use the quadratic formula, it's negative 2p over two. Remember that two in the denominator. So, that's simply negative p.

And, how about the rest? Plus or minus the square root of, now do it in your head. 4p squared minus 4 omega nought squared. So, there's a four in both of those terms. When I pull it outside becomes two. And, the two in the denominator is lurking, waiting to annihilate it.

So, that two disappears entirely, and it will we are left with is, simply, p squared minus omega nought squared.

Now, whenever people write quadratic equations, and arbitrarily put a two in there, it's because they were going to want to solve the quadratic equation using the quadratic formula, and they don't want all those twos and fours to be cluttering up the formula.

That's what we are doing here. Okay, now, the first case is where p is equal to zero. This is going to explain immediately why I wrote that omega nought squared, as you probably already know from physics.

If p is equal to zero, the mass isn't zero.

Otherwise, nothing good would be happening here.

It must be that the damping is zero.

So, p is equal to zero corresponds to undamped.

There is no dashpot. The oscillations are undamped.

And, the equation, then, becomes the solutions, then, are, well, the equation becomes the equation of simple harmonic motion, which, I think you already are used to writing in this form.

And, the reason you're writing in this form because you know when you do that, this becomes the circular frequency of the oscillations. The solutions are pure oscillations, and omega nought is the circular frequency. So, right away from the equation itself, if you write it in this form, you can read off what the frequency of the solutions is going to be, the circular frequency of the solutions.

Now, the solutions themselves, of course, look like, the general solutions look like y equal, in this particular case, the p part is zero. This is zero.

It's simply, so, in this case, r is equal to omega nought i times omega naught plus or minus, but as before we don't bother with the minus sign since one of those roots is good enough.

And then, the solutions are simply c1 cosine omega nought t plus c2 sine omega nought t.

That's if you write it out in the sign, and if you write it using the trigonometric identity, then the other way of writing it is a times the cosine of omega nought t.

But now you will have to put it a phase lag.

So, you have those two forms of writing it.

And, I assume you remember writing the little triangle, which converts one into the other.

Okay, so this justifies calling this omega nought squared rather than k over m.

And now, the question is what does the damp case look like?

It requires a somewhat closer analysis, and it requires a certain amount of thinking. So, let's begin with an epsilon bit of thinking. So, here's my question.

So, in the damped case, I want to be sure that I'm getting oscillations. When do I get oscillations if, well, we get oscillations if those roots are really complex, and not masquerading. Now, when are the roots going to be really complex? This has to be, the inside has to be negative. p squared minus omega squared must be negative.

p squared minus omega nought squared must be less than zero so that we are taking a square root of negative number, and we are getting a real complex roots, really complex roots. In other words, now, this says, remember these numbers are all positive, p and omega nought are positive.

So, the condition is that p should be less than omega nought. In other words, the damping should be less than the circular frequency, except p is not the damping. It's half the damping, and it's not really the damping either because it involved the m, too. You'd better just call it p.

Naturally, I could write the condition out in terms of c, m, and k. So, your book does that, but I'm not going to. It gives it in terms of c, m, and k, which somebody might want to know.

But, you know, we don't have to do everything here. Okay, so let's assume that this is true. What is the solution look like?

Well, we already experimented with that last time.

Remember, there was some guiding thing which was an exponential. And then, down here, we wrote the negative. So, this was an exponential.

In fact, it was the exponential, e to the negative pt. And, in between that, the curve tried to do its thing.

So, the solution looks sort of like this.

It oscillated, but it had to use that exponential function as its guidelines, as its amplitude, in other words. Now, this is a truly terrible picture. It's so terrible, it's unusable. Okay, this picture never happened. Unfortunately, this is not my forte along with a lot of other things.

All right, let's try it better. Here's our better picture.

Okay, there's the exponential. At this point, I'm supposed to have a lecture demonstration.

It's supposed to go up on the thing, so you can all see it.

But then, you wouldn't be able to copy it.

So, at least we are on even terms now.

Okay, how does the actual curve look?

Well, I'm just trying to be fair.

That's all. Okay, after a while, the point is, just so we have something to aim at, let's say, okay, here we are going to go, we're going to get down through there.

Okay then, this is our better curve.

Okay, so I am a solution, a particular solution satisfying this initial condition.

I started here, and that was my initial velocity. The slope of that thing gave me the initial velocity. Now, the interesting question is, the first, in some ways, the most interesting question, though there will be others, too, is what is this spacing? Well, that's a period.

And now, it's half a period. I clearly ought to think of this as the whole period. So, let's call that, I'm going to call this pi over, so this spacing here, from there to there, I will call that pi divided by omega one because this, from here to here, should be, I hope, twice that, two pi over omega one. Now, my question is, so this, for a solution, it's, in fact, is going to cross the axis regularly in that way.

My question is, how does this period, so this is going to be its half period.

I will put period in quotation marks because this isn't really a periodic function because it's decreasing all the time in amplitude. But, it's trying to be periodic. At lease it's doing something periodically. It's crossing the axis periodically. So, this is the half period.

Two pi over omega one would be its full period. What I want to know is, how does that half period, or how does-- omega one is called its pseudo-frequency. This should really be called its pseudo-period. Everything is pseudo.

Everything is fake here. Like, the amoeba has its fake foot and stuff like that. Okay, so this is its pseudo-period, pseudo-frequency, pseudo-circular frequency, but that's hopeless.

I guess it should be circular pseudo-frequency, or I don't know how you say that.

I don't think pseudo is a word all by itself, not even in 18.03, circular.

Okay, here's my question. If the damping goes up, this is the damping term. If the damping goes up, what happens to the pseudo-frequency?

The frequency is how often the curve, this is high-frequency, and this is low-frequency, okay?

So, my question is, which way does the frequency go? If the damping goes up, does the frequency go up or down?

Down. I mean, I'm just asking you to answer intuitively on the basis of your intuition about how this thing explains, how this thing goes, and now let's get the formula. What, in fact, is omega one? What is omega one?

The answer is when I solve the equation, so, r is now, so in other words, if omega one is, sorry, if I have p, if p is no longer zero as it was in the undamped case, what is the root, now? Okay, well, the root is minus p plus or minus the square root of p squared, -- -- now I'm going to write it this way, minus, to indicate that it's really a negative number, omega squared minus p squared.

Now, I'm going to call this, because you see when I change this to sines and cosines, the square root of this number is what's going to become that new frequency.

I'm going to call that minus p plus or minus the square root of minus omega one squared. That's going to be the new frequency. And therefore, the root is going to change so that the corresponding solution is going to look, how?

Well, it's going to be e to the negative pt times, let's write it out first in terms of sines and cosines, times the cosine of, well, the square root of omega one squared is omega one.

But, there's an i out front because of the negative sign in front of that. So, it's going to be the cosine of omega one t plus c2 times the sine of omega one t. Or, if you prefer to write it out in the other form, it's e to the minus p t times some amplitude, which depends on c1 and c2, times the cosine of omega one t minus the phase lag.

Now, when I do that, you see omega one is this pseudo-frequency.

In other words, this number omega one is the same one that I identified here. And, why is that?

Well, because, what are two successive times?

Suppose it crosses, suppose the solution crosses the x-axis, sorry, y-- the t-axis.

For the first time, at the point t1, what's the next time it crosses t2?

Let's jump to the two times across it.

So, I want this to be a whole period, not a half period.

What's t2? Well, I say that t2 is nothing but 2 pi divided by omega one.

And, you can see that because when I plug in, if it's zero, if I have a point where it's zero, so, omega one t minus phi, when will it be zero for the first time?

Well, that will be when the cosine has to be zero.

So, it will be some multiple of, it will be, say, pi over two. Then, the next time this happens will be, if that happens at t1, then the next time it happens will be at t1 plus 2 pi divided by omega one.

That will also be pi over two plus how much?

Plus 2 pi, which is the next time the cosine gets around and is doing its thing, becoming zero as it goes down, not as it's coming up again. In other words, this is what you should add to the first time to get this second time that the cosine becomes zero coming in the direction from top to the bottom.

So, this is, in fact, the frequency with which it's crossing the axis. Now, notice, I'm running out of boards. What a disaster!

In that expression, take a look at it.

I want to know what depends on what.

So, p, in that, we got constants.

We got p. We got phi.

We got A. What else we got?

Omega one. What do these things depend upon? You've got to keep it firmly in mind. This depends only on the ODE.

It's basically the damping. It depends on c and m.

Essentially, it's c over 2m actually. How about phi?

Well, phi, what else depends only on the ODE?

Omega one depends only on the ODE.

What's the formula for omega one?

Omega one squared.

Where do we have it? Omega one squared, I never wrote the formula for you.

So, we have omega nought squared minus p squared equals omega one squared.

What's the relation between them?

That's the Pythagorean theorem. If this is omega nought, then this omega one, this is p.

They make a little, right triangle in other words.

The omega one depends on the spring.

So, it's equal to, well, it's equal to that thing.

So, it depends on the damping. It depends upon the damping, and it depends on the spring constant.

How about the phi and the A? What do they depend on?

They depend upon the initial conditions.

So, the mass of constants, they have different functions.

What's making this complicated is that our answer needs four parameters to describe it. This tells you how fast it's coming down. This tells you the phase lag.

This amplitude modifies, it tells you whether the exponential curve starts going, is like that or goes like this.

And, finally, the omega one is this pseudo-frequency, which tells you how it's bobbing up and down.