18.03SC Practice Problems 12

Gain and phase lag

Solution suggestions:

1. *Explain the notation in the Exponential Response Formula.*

The Exponential Response Formula (ERF) states that a solution to $p(D)x = Ae^{rt}$ is given by $x_p = A\dfrac{e^{rt}}{p(r)}$, as long as $p(r) \neq 0$.

We use D to denote the operation of differentiation with respect to t. $p(D)$ is the total differential operator that acts on the variable x in the differential equation satisfied by x. It encodes the system of the differential equation and gives us the characteristic polynomial $p(s)$. Ae^{rt} is the input to this system, and we are solving for the output x. The ERF says to check whether r, the exponential rate of the input, is a root of the characteristic polynomial (whether $p(r) = 0$), and, if it is not, that we can read off the output x directly as an exponential with the same rate and coefficient as the input, scaled by $1/p(r)$.

2. *Consider the system,*
$$m\ddot{x} + b\dot{x} + kx = ky.$$

In this system, regard $y(t)$ as the input signal and $x(t)$ as the system response. Take $m = 1$, $b = 3$, $k = 4$, $y(t) = A\cos t$. Replace the input signal by a complex exponential y_{cx} of which it is the real part, and use the ERF to compute the exponential ("steady state") system response z_p. Compute H such that $z_p = Hy_{cx}$; H is the complex gain. Find $|H|$ and ϕ such that $H = |H|e^{-i\phi}$. Use this information to compute the gain and the phase lag of the original system. What is the steady state (sinusoidal) solution? Is the amplitude of vibration of the mass greater than or less than the amplitude A of the motion of the far end of the spring?

In this case the equation is $\ddot{x} + 3\dot{x} + 4x = 4A\cos t$, with characteristic polynomial $p(s) = s^2 + 3s + 4$.

A complex replacement for the input signal is $y_{cx} = Ae^{it}$, and the corresponding complex-valued system is $\ddot{z} + 3\dot{z} + 4z = y_{cx} = Ae^{it}$. Here $p(i) = 3 + 3i \neq 0$, so, by the ERF, an exponential response to this complex-valued system is

$$z_p = \frac{4A}{p(i)}e^{it} = \frac{4A}{3 + 3i}e^{it}.$$

From here, the complex gain H can be read off directly as

$$H = \frac{z_p}{y_{cx}} = \frac{4}{3 + 3i} = \frac{2\sqrt{2}}{3}e^{-i\pi/4},$$

which allows us to compute the gain and phase lag of the original (real-valued) system as $|H| = 2\sqrt{2}/3$ and $\phi = \pi/4$, respectively.

The steady state solution of the original system is given by

$$x_p = \text{Re}\,(z_p) = \frac{2\sqrt{2}A}{3}\cos(t - \frac{\pi}{4}),$$

so the amplitude of vibration of the mass here is $2\sqrt{2}A/3$, which is less than A.

3. *Find a solution of $\ddot{x} + 4x = \cos(2t)$. You may have solved similar problems before by guessing a sinusoidal solution and solving for the coefficient. This time solve by making a complex replacement with the appropriate complex exponential on the right hand side and attempt to use an exponential response formula.*

The characteristic polynomial here is $p(s) = s^2 + 4$ and the complex replacement of the equation is $\ddot{z} + 4z = e^{2it}$. Because $p(2i) = 0$, we are at resonance and the standard ERF does not apply. However, $p'(2i) = 4i \neq 0$, so we can apply the Resonant ERF to find that $z_p = \frac{te^{2it}}{4i}$ is a solution to the new complex-valued equation. This means that a solution of the original equation is given by

$$x_p = \text{Re}\,(z_p) = \text{Re}\,\left(\frac{te^{2it}}{4i}\right) = \frac{t}{4}\sin(2t).$$

Note that the factor of i in the denominator of z_p caused the cosine in the real part of the complex exponential to be replaced by a sine in x_p — it gives a phase shift of a quarter-cycle.

Again, you can check this x_p is indeed a solution by making sure that it satisfies the original equation.

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