

## Part II Problems

**Problem 1:** [Linear models] Scrooge McDuck wants to set up a trust fund for his nephew Don. He has fool-proof investments which make a constant interest rate of  $I$ , measured in units of  $(\text{years})^{-1}$  (so  $I = 0.05$  means 5% per year), and he proposes to dole out the money to his profligate nephew at a constant rate  $q$  dollars per year.

(a) Model this process by a differential equation. (Use the symbols  $I$  and  $q$ , rather than specific values for them.) Explain your steps.

(b) Then find the general solution to this differential equation.

(c) Now take  $I = 0.05$ . If Uncle Scrooge wanted to fund the trust so as to provide his nephew with \$1000 per *month* in perpetuity, while maintaining a constant balance in the fund, how much should he invest?

(d) But in fact Uncle Scrooge wants to teach his nephew the virtues of self-reliance, and so plans on having the trust fund run entirely out of money in exactly twenty years. If he wants to give his nephew \$1000 per month, how much should he fund the trust with at the outset? Give the answer to the nearest penny (as Scrooge would insist on).

**Problem 2:** [Solutions to linear equations] Almost all the radon in the world today was created within the past week or so by a chain of radioactive decays beginning mainly from uranium, which has been part of the earth since it was formed. This cascade of decaying elements is quite common, and in this problem we study a “toy model” in which the numbers work out decently. This is about Tatooine, a small world endowed with unusual elements.

A certain isotope of Startium, symbol St, decays with a half-life  $t_S$ . Strangely enough, it decays with equal probability into a certain isotope of either Midium, Mi, or into the little known stable element Endium. Midium is also radioactive, and decays with half-life  $t_M$  into Endium. All the St was in the star-stuff that condensed into Tatooine, and all the Mi and En arise from the decay route described. Also,  $t_M \neq t_S$ .

Use the notation  $x(t)$ ,  $y(t)$ , and  $z(t)$ , for the amount of St, Mi, and En on Tatooine, in units so that  $x(0) = 1$ . Also, assume  $y(0) = 0$  and  $z(0) = 0$ .

(a) Make rough sketches of graphs of  $x$ ,  $y$ ,  $z$ , as functions of  $t$ . What are the limiting values as  $t \rightarrow \infty$ ?

(b) Write down the differential equations controlling  $x$ ,  $y$ , and  $z$ . Be sure to express the constants that occur in these equations correctly in terms of the relevant decay constants. Use the notation  $\sigma$  (Greek letter sigma) for the decay constant for St and  $\mu$  (Greek letter mu) for the decay constant for Mi. Your first step is to relate  $\sigma$  to  $t_S$  and  $\mu$  to  $t_M$ . A check on your answers: the sum  $x + y + z$  is constant, and so we should have  $\dot{x} + \dot{y} + \dot{z} = 0$ .

- (c) Solve these equations, successively, for  $x$ ,  $y$ , and  $z$ .
- (d) At what time does the quantity of Midium peak? (This will depend upon  $\sigma$  and  $\mu$ .)
- (e) Suppose that instead of  $x(0) = 1$ , we had  $x(0) = 2$ . What change will this make to  $x(t)$ ,  $y(t)$ , and  $z(t)$ ?
- (f) Unrelated question: Suppose that  $x(t) = e^t$  is a solution to the differential equation  $t\dot{x} + 2x = q(t)$ . What is  $q(t)$ ? What is the general solution?

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